SOAS Department of Economics

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Working paper
No. 224

June 2019

SOAS University of London

The SOAS Department of Economics Working Paper Series is published electronically by SOAS University of London.

ISSN 1753 – 5816

Suggested citation
Fine, Ben (2019), "A Note on the Relationship between Additive Separability and Decomposability in Measuring Income Inequality", SOAS Department of Economics Working Paper No. 224, London: SOAS University of London.

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A Note on the Relationship between Additive Separability and Decomposability in Measuring Income Inequality

Ben Fine*

Abstract

The purpose of this note is to offer some original technical results in the theoretical measurement of inequality. Whilst most practitioners are content to work with one or other measure, with Gini for example to the fore, and discuss the empirical results that follow from the data, such pragmatism involves a certain degree of arbitrary ethical judgement over how more for one rather than another should be assessed. At a deeper level of principle, constructing measures of inequality proceeds by specifying conditions like homogeneity for which multiplying all incomes by a common factor should leave a measure unchanged.

Such conditions are the starting point for this contribution, drawing upon a rich literature that already exists. More specifically, this note explicitly explores the relationships between additive separability and homotheticity of measures of welfare (closely related to derived measures of inequality), and homogeneity and decomposability in direct measures of inequality, drawing upon the previous literature along the way to make this possible. An interrogation is made of the resonances and dissonances between the classic contributions of Atkinson (1970) and Shorrocks (1980). In brief, in the presence of otherwise common assumptions, it is shown that additive separability and homotheticity of welfare are stronger combined conditions than decomposability and homogeneity of income inequality. The gap between the two, however, can be closed by adding an extra term around total income to the measure of welfare.

Keywords: inequality measurement

JEL classification: D63, D6

Acknowledgements: Thanks to Pedro Mendes Loureiro for comments and continuing discussions, and for correspondence with Tony Shorrocks. The usual disclaimers apply.

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1. Introduction

The purpose of this note is to offer some original technical results in the theoretical measurement of inequality. Whilst most practitioners are content to work with one or other measure, with Gini for example to the fore, and discuss the empirical results that follow from the data, such pragmatism involves a certain degree of arbitrary ethical judgement over how more for one rather than another should be assessed. At a deeper level of principle, constructing measures of inequality proceeds by specifying conditions like homogeneity for which multiplying all incomes by a common factor should leave a measure unchanged.

Such conditions are the starting point for this contribution, drawing upon a rich literature that already exists. More specifically, and in addition, it is concerned with two separate, but closely-related, ways of measuring inequality. One is to start with a measure of society's (income-derived) welfare and deduce implications for inequality from this. Common conditions to impose in measuring welfare are symmetry (different individuals should be treated the same), additive separability (welfare derives from aggregating over that of individuals), and homotheticity (welfare trade-offs between individuals are the same at different levels for the same welfare ratios – hard to say but easy to put down mathematically, see below). A second, more direct way to go about measuring inequality is to impose condition on such measures themselves. Here, common conditions are homogeneity, as already mentioned, and decomposability (that we should be able to disaggregate our measure of inequality into within and between group components).

On the face of it, these two ways of proceeding are closely related but remain different. Strangely, the relationship between the two has not been closely studied. Do they give the same or different results and why? As a result, this note explicitly explores the relationships between additive separability and homotheticity of measures of welfare (closely related to derived measures of inequality), and homogeneity and decomposability in direct measures of inequality, drawing upon the previous literature along the way to make this possible. More specifically, an interrogation is made of the resonances and dissonances between the classic contributions of Atkinson (1970) and Shorrocks (1980). In brief, in the presence of otherwise common assumptions, it is shown that additive separability and homotheticity of welfare are stronger combined conditions than decomposability and homogeneity of income inequality. The gap between the two, however, can be closed by adding an extra term around total income to the measure of welfare.

2. From Atkinson …

A classic example of the first approach to measuring inequality is provided by Atkinson (1970). He assumes that social welfare is additively separable. This means that for two individuals with income x and y, and symmetry between individuals,[1] welfare W is given by:[2]

$W(x, y)=f(x)+f(y)$ for some function f.

This might be justified in the context of concern with income alone, isolated individuals, and no reason available other than to sum individual welfares in assessing overall welfare. In addition, Atkinson assumes that the measure for W is homothetic, meaning, as mentioned, that the rate at which W changes between the individuals is invariant to the ratio x/y. Homotheticity is a weaker condition that homogeneity with the scale at which W changes not fixed by some degree of homogeneity.[3] Nonetheless, it makes sense for W at least to increase with t in the equation below when individuals will have higher incomes.

Homotheticity implies that:

$f^{/}(x)/f^{/}(y)=f^{/}(tx)/f^{/}(ty)$ for all t

Cross-multiplying, differentiating for t, setting t=1, and rearranging gives:

$xf^{//}(x)/f^{/}(x)=yf^{//}(y)/f^{/}(y)= c-1$, say, a constant.

With $xf^{//}(x)+(1-c)f^{/}(x)=0$, it follows that:

$x^{1-c}f^{//}(x)+(1-c)\ x^{-c}f^{/}(x)=0$

Integrating gives:

$x^{1-c}f^{/}(x)=A$ for some first constant of integration, A, from which it follows:

$f(x)=Ax^{c}+B$ for $c \neq 0$ (replacing A by A/c)

and f(x)=Alogx+B for c=0 otherwise, leaving A unchanged, and for second constant of integration B.

For inequality aversion it makes sense to confine c between 0 and 1. Note then that the maximum measure of welfare is when total income x+y is equally shared and there is perfect equality, with $W=2\{(x+y)/2\}^{c}= 2^{1-c}(x+y)$, and the minimum is perfect inequality with one individual getting all of x+y, with $W=(x+y)^{c}$.[4] If we also want to standardise our measure for overall income and for it to lie between 0 (perfect equality) and 1 (perfect inequality), and to expand the number of individuals to n, the measure for inequality becomes, with μ_x mean income for the x_i:[5]

$\{n^{1-c}-\Sigma(x_i/n\mu)^{c}\}/\{n^{1-c} - 1\}$

If the measure is merely restricted to 0 as minimum but without maximum, the measures become, leaving aside c=0

$\{n^{1-c}-\Sigma(x_i/n\mu)^{c}\}$ which can also be written as:[6]

$\{1/n^{c}\mu^{c}\}\Sigma(\mu^{c}-x_i^{c})$

Note that this has been rendered homogeneous of degree zero in incomes although starting off as homothetic in (additively separable welfare). If we increase all incomes, but with these remaining in the same proportions, the measure of inequality remains the same even though individual and overall welfare increases with f, albeit tempered by the inequality aversion parameter c. If we reverse the sign, so perfect equality gives a zero measure, this is:[7]

$\{1/n^{c}\mu^{c}\}\Sigma(x_i^{c}-\mu^{c})$

3. … To Shorrocks

What this reveals is just how powerful are the principles of additive separability and homotheticity in restricting the functional forms that can be taken in measuring overall welfare, with corresponding implications for inequality. The same follows from the other approach, addressing inequality directly as illustrated by Shorrock's (1980) classic piece in which the target is one of decomposing measures of inequality into within group and between group contributions. This involves more taxing technical complexities and so the reader is referred more than for the account of Atkinson to the original contribution for the details.[8]

Let **x** be a vector of incomes for a population of size n. Let the population be reordered at will (a way of defining symmetry), and partition the population into subgroups of vectors of incomes $\mathbf{x_1}$, $\mathbf{x_2}$, …, $\mathbf{x_m}$, in m groups, each of size n_i, with $n=\Sigma n_i$. Decomposability is essentially defined by Shorrocks to mean our measure of inequality, I, takes the following form for any partition of **x**, where μ_i are partition group means and w_i weights:

$$I(\mathbf{x}, n_x) = \Sigma\, w_i\, I(\mathbf{x_i}, n_i)+I(\mu_1, \mu_1, \dots \mu_1, \mu_2, \dots \mu_2, \dots \mu_m, n)$$

Decomposabililty simply means we can disaggregate overall inequality across any set of subgroups of the incomes into two parts. One is to take a weighted, w_i, combination of the inequalities from the inequality measures of the sub-groups. The other is the inequality measure as if each subgroup had equally distributed mean of its income. Taking I to be lower bound by zero for perfect equality and assuming I is symmetric in the incomes, without any other assumptions, Shorrock shows that I necessarily takes the form (when not decomposed), as follows:[9]

$$I(\mathbf{x}, n)=\{1/\theta(\mu, n)\}\Sigma\{f(x_i)-f(\mu)\}$$

with f strictly convex (positive but diminishing returns) and where x_i runs over the whole of the $\mathbf{x_i}$ for which the overall mean is μ, and θ is essentially a scaling factor for the overall mean, μ, and the number of individuals, n.[10]

It is also shown that the weights for decomposing are fixed by the very same θ function of the subgroup mean and size, scaled by the overall population:

$$w_i=\theta(\mu_i, n_i)/\theta(\mu, n)$$

Once again, as with Atkinson, these are remarkably powerful results in restricting the functional forms that can be taken by I (and, it should be emphasised the weights, w). It is immediately apparent, though, that, in the presence of symmetry, the Shorrocks measure can be interpreted as additively separable in welfare for given μ and n (serving, through f and θ as scaling factors, in converting inequality to welfare).

In addition, Shorrocks is only more general than Atkinson because the latter assumes homotheticity of the welfare function. This, however, is equivalent to homogeneity for the Atkinson inequality measure, as shown towards the end of the Atkinson discussion of the previous section[11] If homogeneity of the inequality measure, as is assumed for Shorrocks, gives the same measures as for Atkinson, it would immediately follow that Atkinson and Shorrocks are simply equivalent to one another in the presence of symmetry even though differently derived in motivation and assumptions.

This, however, is untrue as is implicitly shown by Shorrocks. Following his derivation of I in assuming it is homogeneous in incomes, then:[12]

$\{1/\theta(\mu, n)\}\Sigma\{f(x_i)-f(\mu)\}=\{1/\theta(t\mu, n)\}\Sigma\{f(tx_i)-f(t\mu)\}$, for all t.

Dropping the n within θ for convenience, differentiating for t, and setting t=1, gives after some rearrangement:

$\Sigma[\theta\{x_i f^{/}(x_i)-\mu f^{/}(\mu)\}-\theta^{/}\{f(x_i)-f(\mu)\}]=0$

Differentiating this again once each for both x_i and x_j and subtracting, one from the other gives:

$\{x_i f^{//}(x_i)- x_j f^{//}(x_j)\}-c\{f^{/}(x_i)-f^{/}(x_j)\}=0$ where $c= \mu\theta^{/}(\mu)/\theta(\mu)$

c must be a constant as independent of x_i and x_j as such, since each of these can be varied whilst keeping μ fixed by varying the other; hence the derivation of the equation. As a result, it also follows that $\theta(\mu)=K\mu^c$ for some constant K.

In addition, $xf^{//}(x)-cf^{/}(x)=b$ for some constant b, since it follows that:

$x_i f^{//}(x_i)-cf^{/}(x_i)=x_j f^{//}(x_j)-cf^{/}(x_j)$

The solutions to the differential equation for f are:

$f(y)= Ay^c/\{c(c-1)\}+By+C$, when $c\neq0,1$

$f(y)=-A\log y+By+C$, c=0

and $f(y)=Ay\log y+By+C$, c=l:

This is remarkably similar to Atkinson's functional forms but is more general, with the last extra functional form and B≠0 for each of the other functional forms.

4. And Beyond

So, given symmetry and homogeneity of the inequality function, I, additive separability is a stronger condition than decomposability since it allows for (remarkably) fewer functional forms. But what exactly is it that makes the two conditions different? In terms of the solutions from the differential equations, the answer is straightforward. Shorrocks' equation becomes Atkinson's if and only if b=0 for Shorrocks' equation, $xf^{//}(x)-cf^{/}(x)=b$. For b=0, $xf^{//}/f^{/}$ is zero, or the elasticity of the rate at which individual welfare increases with income is constant, not particularly open to intuitive understanding (as opposed to its being variable for b≠0).

But, going back to decomposability, its measure of welfare, as observed, is readily seen to be the additively separable sum of individual welfares **minus** the welfare that would accrue if each benefitted from the mean income. This might be justified on the grounds that as income increases, and basic needs are met, then dealing with inequality becomes relatively more important. On the other hand, though, it could be argued that when income is in short supply, meeting basic needs, and greater equality, is more imperative. In technical terms, it is a matter of whether income inequality is an ethical necessity or a luxury!

Be all this as it may, for given n, the Atkinson measure for welfare can be generalised to the following form, in subtracting a function of total income:

$W=\Sigma\{f(x_i)-g(\Sigma x_i)\}$

where g is a function of total income and takes the form of the special case of $f((\Sigma x_i/\mathbf{n})$ for Shorrocks. If, as before, W is assumed to be homothetic, let $x_i=0$, apart for x_1 and x_2, and denote these by x and y, respectively. Then:[13]

$\{f^{/}(x)-g^{/}(x+y)\}/\{f^{/}(y)-g^{/}(x+y)\}$ is a function, h(x/y) for some h.

Replace $f^{/}$ and $g^{/}$ by f and g not forgetting each has been substituted for by its differential. Then, for homotheticity:

$h(x/y) = 1+ \{(f(x)-f(y))/\{f(y)-g(x+y)\}$

Let x=ay and so $\{f(ay)-f(y)\}/\{f(y)-g((a+1)y)\}=1/C(a)$ for some function C.

Then:

$\{f(ay)-f(y)\}C(a)=\{f(y)-g((a+1)y)\}$

For a=0, $\{f(0)–f(y)\}C(0)=\{f(y)-g(y)\}$, so

$g(y)=-f(0)C(0)+(1+C(0))f(y)=Af(y)+B$, say, where $A=1+C(0)$ and $B=-f(0)C(0)$.

Substituting gives:

$\{f(ay)-f(y)\}C(a)=\{f(y)-Af((a+1)y)-B\}$

First, differentiate for y:

$(af^{/}(ay)-f^{/}(y))C(a)=f^{/}(y)-A(a+1)f^{/}((a+1)y)$

For y=0, this gives:

$(a-1)f^{/}(0)C(a)= (1-Aa-A)f^{/}(0)$, so $f^{/}(0)=0$ or $C(a)=\{A(a+1)-1\}/(1-a)$.

Now differentiate from above for a:

$\{f(ay)-f(y)\}C^{/}(a)+\{yf^{/}(ay)\}C(a)=-yAf^{/}((a+1)y)$

For a=0:

$\{f(0)-f(y)\}C^{/}(0)+\{yf^{/}(0)\}C(0)=-yAf^{/}(y)$

For $C(a)=\{A(a+1)-1\}/(1-a)$, substitute back into the definition of C, replace a by x/y, rearrange and we find:

$g(x+y)=\{A(x+y)-1\}/\{(x-y)(f(x)-f(y))\} + f(y)$

Swapping x and y, and comparing, leads to f(x)=f(y), as large first term remains the same. This is a trivial solution of constant for f.

Otherwise for $f^{/}(0)=0$

$\{f(0)-f(y)\}C^{/}(0)=-yAf^{/}(y)$

Hence:

$yf^{/}(y)-cf(y)=b$ where $c=C^{/}(0)/A$ and $b=-C^{/}(0)f(0)/A$.

Recalling f displaced $f^{/}$, it follows that f at least is a Shorrocks-type functional form. In addition, as g=Af+B before such displacement, it follows that restored $g(x)=Af(x)+Bx+C$, for constants A, B and C.

B and C should not worry us too much as Shorrocks' f is itself only unique up to addition of linear terms. On the other hand, our amendment of Atkinson is more general than for Shorrocks (for which A=1). Thus, as is readily verified, A can vary freely, whether positive or negative, to satisfy the technical conditions depending on how much the welfare of total income is to count for or against that of aggregate individual welfares derived from that income.

In short, the Atkinson measure becomes $W=\Sigma\{f(x_i)-Af(\Sigma x_i)-B\Sigma x_i -C\}$, for constants A, B and C, where f only takes the functional forms specified by Shorrocks (but not restricted to Atkinson). If, however, we force f and g to be the same at the outset, and define Atkinson welfare in the more restricted form, $W=\Sigma\{f(x_i)-f(\Sigma x_i)\}$ as homothetic, then it follows that this is equivalent to decomposability and homogeneity of the corresponding inequality index.

5. Concluding Observations

It has been shown, in the presence of symmetry, that the conditions of homotheticity of additively separable welfare (Atkinson) and of homogeneity of decomposability of inequality measures (Shorrocks) are close to, but not identical with, the latter allowing for more generality than the former in functional forms. Shorrocks derives from solutions to the differential equation:

$$xf^{//}(x)-cf^{/}(x)=b$$

for which b=0 for Atkinson. However, the latter becomes more general in case, like Shorrocks, a term for total income is added to aggregate welfare as is implicit in the Shorrocks' measure of inequality, although the greater generality is limited to a linear function of the additive individual welfare function. Atkinson and Shorrocks become equivalent if and only if the measure of individual welfare and total income is confined to the same function.

These results seem to offer some clumsy satisfaction in terms of completeness of the relationship between the two measures and the two sets of properties involved. Intellectual tidiness aside, this may all seem to be or limited importance. They do, however, have significant implications once the common condition of symmetry is relaxed, especially in addressing issues of within and between group measurement of inequality at which decomposability is targeted. The benefits of forging equivalence between the two measures is that conditions on one can be strengthened, or restricted, with corresponding implications teased out for the other. Sometimes it is easier working with asymmetries in the Atkinson framework, sometimes in Shorrocks, depending on who and how we want to treat groups differently. But this will all be the topic of a separate contribution.

Endnotes

[1] For Atkinson without symmetry, see Fine (1985). It merely results in the replacement of each f by $b_i f$, for constants b_i, i.e. the functional form does not change showing what a powerful condition is additive separability with or without symmetry.

[2] Throughout, where possible, we deal in two individuals/incomes alone to ease presentation and understanding.

[3] Homotheticity is usually defined in terms of a monotonic increasing function of a homogeneous function.

[4] Leaving aside, A and B which serve as scaling factors, see below.

[5] For c=0, the measure becomes $\{\Sigma(\log x_i - \log\mu)\}/\{\log n - (n-1)\log\mu\}$.

[6] Leaving aside, scaling factor B, and for c=0, $\Sigma(\log\mu - \log x_i)$.

[7] And $\Sigma(\log x_i - \log\mu)$, notably independent of n.

[8] Unfortunately, we cannot work with just two individuals as previously, as the concern is with between and within group inequalities for which we need numbers of individuals.

[9] There are also some minor technical assumptions which we do not cover, for Atkinson as well as for Shorrocks, not least to exclude lexicographic orders and the like for lack of continuity and differentiability of the measures and functions.

[10] The proofs are made by forging equalities across cleverly chosen distributions of incomes, partitions, changes in individual incomes, possibly at the margins (i.e differentiating changes), through reliance upon decompositions and symmetry. Thus, informally for example, I(x, x, y) would equal wI(x, y)+I(x, (x+y)/2, (x+y)/2), using decompositions into (x) and (x, y) and given that I(x)=0.

[11] As a Shorrocks function, see above, Atkinson has $f(x)=x^c$ and $\theta(\mu,n)=n^c\mu^c$, and similarly for the case c=0.

[12] It is possible to skip the derivation and go to the resulting functional forms towards the end of the section, although the derivation sheds some light on where the two methods differ.

[13] As before, the derivations can be skipped and the reader jump to final paragraph of this section to see the functional forms for the Atkinson measure.

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