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Mathematical Economics and Control Theory: Studies in Policy Optimisation

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Abstract

Chapter 1 deals with the origin and limitations of mathematical economics and its implications for economic applications of optimal control theory. Using an historical approach, we have proposed a hypothesis on the origin and limitations of classical and modern mathematical economics. Similar hypotheses proposed by Cournot, Walras, von Neumann-Morgenstern and Debreu are shown not to be convincing. Conditions are established under which applications of mathematical methods, in general, and optimal control theory, in particular, may produce economic results of value.

Chapter 2 concerns the formation and development of optimal control applications to economic policy optimisation. It is shown that the application of *mathematical* control theory (as compared with *engineering* control) may significantly contribute to *mathematical economics* (as compared to *econometrics*). The development of optimal growth theory has been examined as an example. Within the context of economic policy optimisation, a critical examination of the recent developments in macroeconomic modelling, the relationship between theory and observation, rational expectations, the Lucas critique and the problem of time-inconsistency is presented.

Chapter 3 provides the first illustration of the main theme of the earlier chapters. Using the generalised Hamiltonian in Pontryagin's maximum principle, as well as using Bellman's dynamic programming, we have obtained a number of new results on the mathematical properties of optimal consumption under liquidity constraints. For example, we have demonstrated how the response of optimal consumption to liquidity constraints is conditioned by the consumer's intertemporal elasticity of substitution. Considered as a mathematical structure, this is shown to capture the effects of the following variables on the optimal consumption path: pure preference parameters, the interest rates variations and the structural parameters prevailing in the credit markets.

In chapter 4, the dynamic Leontief model, which according to the conditions established in chapter 1, is one of the most successful applications of mathematical methods to economic policy analysis, is first considered as a control problem. We have then obtained the optimal consumption path for deterministic and stochastic dynamic Leontief models with substitute activities which are in turn formulated in deterministic and stochastic environments. Our solution uses Pontryagin's maximum principle, Bellman's method and Åstrom's Lemma on stochastic dynamic programming.

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Chapter One

The Origin and Limitations of Mathematical Economics and its Implications for Economic Applications of Optimal Control: An Historical Approach

1.1 Introduction

Our starting point is the recognition of the fact that optimal control is no more than an advanced mathematical method in the field of dynamic optimization theory. Its applications to economic analysis are therefore constrained by the limitations in mathematical treatment of economics. It follows that many questions on economic applications of optimal control cannot successfully be examined without a direct reference to the origin and limitations of the mathematization of economics. As an example, consider the following questions: What is the logical justification for optimal control applications to economic analysis? What are the salient features of optimal control theory which have made it so attractive to the community of mathematical economists and econometricians? What factors have contributed to its successful applications and what have been the underlying conditions responsible for its partial failure in satisfying the earlier great optimism? The key to all these questions lies in the mathematical nature of optimal control theory and on its capabilities and limitations in handling specific problems in economic analysis.

We start the analysis of the origin of mathematical economics by presenting the following questions in section 1.2: Why is Cournot (1838) unanimously agreed as the birth of mathematical economics while 38 research work on this subject had been published before that? Why was the mathematical economics of Cournot totally ignored by classical economists for more than 30 years until Jevons (1871) revived it? Why is the importance and significance of applications of mathematics to economics still an unsettled issue whereas physical sciences can hardly do without mathematics? To answer these questions, we have first examined the hypotheses put forward by Debreu, Cournot, Walras and von Neumann-Morgenstern in sections 1.2.1 to 1.2.4, respectively. We have found that none of these hypotheses is satisfactory.

Our hypothesis of one-dimensionalisation of economic analysis is discussed in section 1.2.5. Historically, advances in classical economics together with theoretical developments in Marxian economics had produced a number of different economic doctrines which were considered by the advocates of mathematical economics as a *chaotic state.* The "scientific" or mathematical approach and the socio-political approach as two possible responses to such an environment of multi-dimensionality in economic studies are examined. Mathematical economics and the formation of economic science is discussed in section 1.2.5 where "the nature of classical mathematical economics" and "mathematical economics as a remedy to multi-dimensional political economy" are analysed. The hypothesis that mathematical economics has been developed as a response to Marxian economics is another topic which is discussed in section 1.2.5.

The origin and formation of modern mathematical economics is the subject of section 1.3. The origin and nature of forty years recession in theoretical developments in mathematical economics from Marshall(1890) to the emergence of a new era in mathematical economics are explained in this section. Debreu's hypothesis which regards the publication of von Neumann and Morgenstern's *Theory of Games and Economic Behaviour* (1944) as the starting point of modern mathematical economics is critically examined and rejected in section 1.3.1. In section 1.3.2, we have proposed our hypothesis that the creation of the *Econometric Society* in 1930, the Cowles Commission in 1932 and the concomitant advances in coordinated research programmes in mathematical economics can be considered as epoch-making events which have marked the beginning of modern mathematical economics.

To identify the salient features of modern mathematical economics, we have compared the attitudes of classical mathematical economists and their perceptions on the limitations of economic applications of mathematical methods with those of modern mathematical economists in sections 1.4.1 and 1.4.2, respectively. The theoretically important question of whether economic truths are discoverable through the instrumentality of mathematics is discussed in section 1.4.2.

The objectives of the Cowles Commission and the Econometric Society in coordinating the mutual penetration of economic theory, mathematical methods and statistics, discussed in section 1.3.2, have been seriously challenged by the recent developments of the relationship between mathematical economics and econometrics. This problem is discussed in section 1.5 where the nature of the disparity between mathematical economics and econometrics in building up models for empirical analysis is discussed. This provides a background for section 2.5 in Chapter two where a discussion of the critique of macro-econometric models is presented in the light of the ongoing debates on the relationship between theory and observation.

Having discussed the origin of mathematical economics, the salient features of modern mathematical economics and its limitations and the problems associated with the relationship between mathematical economics and econometrics, the economic applications of optimal control theory is discussed in section 1.6 as an illustration. In section 1.7 attempts are made to identify the sources of limitations in the mathematization of economics. The implications of such limitations are also studied for economic applications of optimal control. It is hypothesised that the logic of abstraction employed in obtaining basic economic concepts of narrow components which facilitate quantitative formulations plays the key role in generating such limitations. Conditions under which economic applications of optimal control can produce more reasonable results are also discussed in this section. Finally a summary and concluding remarks are presented in section 1.8.

1.2 Mathematization of Economics: Hypotheses on the Origin and Significance of Mathematical Economics

We present our argument in an historical context. This will, hopefully, provide a basis for future speculations. It is now agreed that Civa (1642-1734), an Italian mathematician, is the first author to apply mathematical methods to economic problems. His work on money, written in 1711, is the first true example of mathematical economics in which the ideas of definitions, postulates, remarks, propositions, theorems and corollaries are used in analysing money. This work, however, was completely ignored until 1871 when it appeared in Jevons's *List of Mathematico-Economic Books, Memoirs, and Other Published Writings.*

127 years after Civa's work, Cournot, professor of mathematics at Lyon and the Rector of the Academy of Grenoble, published his epoch-making contribution to economics under the title *Recherches sur les Principes Mathématiques de la*

Théorie des Rechesses in 1838. Economists today unanimously agree that the symbolic birth of mathematical economics is the year in which Cournot published his book. The first key question is that why Cournot (1838) and not Civa (1711) or any other work among the 38 research work published before Cournot on mathematical economics,[1] is not considered to be the pioneering work in this field? What has made Cournot's work to be recognised as an epoch-making contribution? Has Cournot's mathematical excellence been responsible for this success or has it been realized that this work can be regarded as starting point for a new current of thoughts in political economy? We will come back to these questions in section 1.2.5.

However, Cournot's book received little or no attention at the time: "For several years not a single copy of the book was sold. In 1863 the author tried to overcome the indifference of the public by recasting the work and omitting the algebraic formulae. This time the book was called *Principes de la Théorié des Richesses.* In 1876 he published it again in a still more elementary form and under the title of *Revue Sommaire des Doctrines Économiques* but with the same result".[2] J. B. Cherriman, a Canadian mathematician, published a ten page review on Cournot in 1857. This was the only published recognition of Cournot's book. Cournot's significant contribution to mathematical economics was finally revived by Jevons (1871). On page 26 in the preface, Jevons stated that "This work must occupy a remarkable position in the history of the subject. It is strange that it should have remained for me among Englishmen to discuss its value".[3]

According to Fisher (1891, p.109) "The introduction of mathematical method

[1]For a list of 38 work before Cournot, i.e. during the period 1711-1838 and 62 work from Cournot to Jevons, i.e. 1838-1871, published on mathematical economics, see Jevons's *List of Mathematico-Economic Books, Memoirs and Other Published Writings*, pp. 322-339, in his *Theory of Political Economy*, 1871.

[2]See Charles Gide and Charles Rist (1909, 1948), p. 499.

[3]All references to Jevons (1871) made in this chapter are from its 4th edition, London: Macmillan, 1911, 339 pages.

marks a stage of growth -perhaps it is not extravagant to say, the entrance of political economy on a scientific era ... Before Jevons all the many attempts at mathematical treatment fell flat. Every writer suffered complete oblivion until Jevons unearthed their volume in his bibliography". This will lead us to the second key question: Why Cournot's significant contribution together with previous work on mathematical economics were completely ignored, or were not taken seriously, by classical economists?

A more fundamental point which is related to the above mentioned two key questions is the following: Why has the application of mathematical methods in economics not been very successful? In other words, if physical sciences can hardly develop without mathematics why is the importance and significance of mathematics in economic analysis not yet a settled question? We examine these points in the context of the following hypotheses in sections 1.2.1-1.2.5.

1.2.1 Debreu's Incidentality Hypothesis of Early Developments in Mathematical Economics

The hypothesis that mathematical economics has emerged from nowhere and has grown with no aims while being independent of any current of economic thoughts has received supports from a number of economic historians and even from mathematical economists. Gherity (1990) held that "For many years historians of economics saw those who pioneered the application of mathematics to economics as individuals who had appeared out of nowhere, spoken their piece and fallen back into oblivion without impact or influence on their contemporaries or on those who came after". In a similar but more elaborated line of argument, Debreu (1986, p. 1259) regarded the emergence of mathematical economics simply as an historical coincidence: "[The early progress of mathematical economics] is marked by several major scientific accidents. One of them occurred in 1838 ... with the publication of Augustin Cournot's [book] ... The University of Lausanne was responsible for two others of those accidents. When Léon Walras delivered his first professional lecture there on 16 December 1870, he had held no previous academic appointments; he had published a novel and short story[4] but nothing on economic theory and he was exactly 36 ... For Vilfredo Pareto, who succeeded Walras in his chair in 1893, it was also a first academic appointment; like his predecessor he had not published anything on economic theory before; and he was 45".[5]

The hypothesis of incidentality of developments in mathematical economics has also been reported in Robertson (1949, p. 535). He speaks of mathematical economists as "more or less isolated figures who cannot be said to have contributed to a *current* of thought because there is no discernible *flow*". Theocharis (1983, p. 1) has tried to partially improve this hypothesis by saying that "in many instances

[4]In here, Debreu has referred to *Francis Sauveur*, published by Walras in 1858, Paris: E. Dentu.

[5]See, also, Debreu (1987, p. 399). It should be noted that there are a number of errors in Debreu's (1986) comments on Walras and Pareto. The following facts reveal such shortcomings: 1) Walras, unhappy with his engineering studies at *Ecole des Mines* and dissatisfied with literature and journalism as his second academic challenge, was persuaded by his father, an economist, to study economics at the age of 24 to continue his father's research on mathematical economics. It was after 12 years of hard work that this self-taught economist was offered the new chair of political economy at the University of Lausanne. 2) The reason that Walras had held no previous academic position was his lack of any officially recognised educational credentials in economics. 3) Walras presented a paper on Taxation in 1860 in an international conference on taxation in Lausanne which remarkably impressed the audience. [For 1, 2 and 3 see Jaffé (1954), the translator's forward to Walras (1874), pp. 5-6]. 5) During 1859 to 1862, when Walras was working as a journalist for the *Journal des Economistes* and *La Presse*, he published *l'Economie Politique et le Justice*, Paris: Guillaunin, 1860, in which he strongly attacked the normative economic doctrines of P.-J. Proudhon, [See Donald A. Walker (1987), p. 852]. 6) Vilfredo Pareto, graduated in mathematical and physical sciences in 1867 and engineering in 1870, started to write and publish articles, as early as 1872, on commerce, the state of Italian industry, railways, advantages and disadvantages of public and private use of the railway system and support of free trades to prevent any form of state intervention in economic activity. Pareto was one of the founders of the Adam Smith Society, which spread and upheld the doctrine of economic liberalism. In October 1891, Pareto published his controversial article "L'Italie économique" which was followed by another critical work in April 1892 on Italian Government economic policies. In 1890, Maffeo Pantaléoni, the famous Italian economist, advised him to study the work of Walras on mathematical economics, and Pareto met Walras himself on September 1890, before accepting the chair of Walras in political economy in 1893. [See G. Busino (1987), p. 800]. The above facts clearly reject Debreu's hypothesis on the incidentality of mathematical economics in Lausanne school.

these writers were familiar with the work of their predecessors and did, in fact, build upon them".

We can classify the pre-Cournot's early mathematical economists, starting from Civa (1711), as purely academic and intellectual exercises in which economic concepts were being translated into mathematical symbols and operations. These work all lacked any sense of direction. On the contrary, pioneers in mathematical economics in the 19th century, i.e. Cournot (1838), Jevons (1871), Walras (1874), Marshall (1890), Fisher (1891) and Pareto (1896), were all completely aware of their backgrounds, their current positions and, most important, their aims. In this section, we establish the validity of our hypothesis for Cournot, which is more controversial due to his historical isolation of being 30 years before Jevons. In the course of our future analysis, the validity of this hypothesis will be established for Jevons, Walras and Pareto.

To reject Debreu's hypothesis of incidentality of Cournot's contribution we refer to the first paragraph of the preface in Cournot (1838). He has clearly admired and appreciated the one hundred years of developments in political economy before him, but at the same time has urged the necessity of developing a *positive economics* due to the fact that the public has become so *tired* of theories of different economic *systems and doctrines*: "The science known as Political Economy, which for a century has so much interested thinkers, is to-day more generally diffused than ever before. It shares with politics proper the attention of the great journals, which are to-day the most important means of spreading information; but the public is so tired of theories and systems that now the demand is for so-called "positive" matters, i.e. in political economy, custom-house abstracts, statistical documents and government reports, such as will throw the light of experience on the important questions which are being agitated before the country and which so greatly interest all classes of society".[6]

Despite the fact that Cournot's prime objective was to support an econometric type analysis, the shortages of organised data and the lack of appropriate statistical methods of estimation, forced him to concentrate on pure theorization of economic concepts towards building up a positive economics. The second paragraph in his preface explains this point: "I will only observe that *Theory* ought not to be confounded with systems ... and that, to a man of my position in particular, more than to any other, it should be permissible to consider from an exclusively theoretical standpoint, a subject of general interest which has so many different sides".

The few facts presented above reject the hypothesis that mathematical economics has come from nowhere and has developed with no clear aims. But the question remains why the early mathematical economists failed to achieve their objectives? Or equivalently, why eminent classical economists did not employ mathematical methods in their economic analysis? In this regard, we examine the following hypothesis.

1.2.2 Cournot's Hypothesis: Erroneous Presentations and Poor Mathematical Knowledge

According to this hypothesis, the inaccurate early writings in mathematical economics and their weak economic contents together with the fact that the community of classical economists were not well equipped with basic mathematical knowledge, were the significant factors which hindered the pace of developments in mathematical economics. This hypothesis which was first proposed by Cournot (1838) has received support from a number of economists including Fisher (1891).

[6]All references to Cournot (1838), made in this Chapter, are from its English translation by Nathaniel T. Bacon: *Researches into the Mathematical Principles of the Theory of Wealth*, New York: Macmillan, 1897, reprinted 1927.

According to Cournot "The attempts which have been made in this direction have remained very little known and I have been able to learn only the titles of them, except one, *Les Pricipes de l'Economie Politique* by Canard, a small work published in 1801 and crowned by the *Institut.* These pretended principles are so radically at fault and the application of them is so erroneous, that the approval of a distinguished body of men was unable to preserve the work from oblivion. It is easy to see why essays of this nature should not incline such economists as Say and Ricardo to algebra" [Cournot (1838), preface].

If the above hypothesis was true, then Cournot's work, a concise, original and well presented work on mathematical treatment of economics, should have attracted the attention of economists of his time; but we know that his work was absolutely ignored by economists for more than 30 years until Jevons (1871) revived it. The subsequent developments in mathematical economics can also provide useful evidence to reject Cournot's hypothesis. For example, the concise and mathematically elaborated contribution of Walras, i.e. *Elements d'Économie Politique Pure* was hardly noticed in France during the twenty-five years after its publication in 1874. Even Alfred Marshall, a mathematician and an economist, has only mentioned Walras in the briefest of comments in his *Principles of Economics* (1890) and did not take Walras's general equilibrium seriously at all. Eighty years after Walras, mathematical economists of the 20th century such as Abraham Wald, John von Neumann, John Hicks, Frank Hahn, Oscar Lange, Paul Samuelson, Lionel McKenzie, Gerad Debreu, Kenneth Arrow and Michio Morishima acknowledged Walras's contribution and paid attention towards further developments in Walras's general equilibrium analysis.[7]

Let us now examine the hypothesis of poor mathematical knowledge. According to this hypothesis the complete oblivion of early developments in mathe-

[7] See Weitraub (1986) for extensions of Walras's general equilibrium.

matical economics was mainly due to the inadequacy of mathematical knowledge among economists. This argument is not satisfactory either. Despite the fact that economists, in a rare unanimous agreement, would select Jevons (1871), a mathematician, a logician and educated in chemistry but self-taught in economics, as the first economist who made known to the world the remarkable position of Cournot in the history of economics, Jevons himself confessed that he could not mathematically understand all parts of Cournot's book. On page *xxx* in the preface to the 2nd edition of his book (1879), Jevons maintained that "Even now I have by no means mastered all parts of it, my mathematical power being insufficient to enable me to follow Cournot in all parts of his analysis".

The above quotation raises the question that if Jevons, like many other economists before him, did not completely understand Cournot, what made him pronounce the forgotten Cournot as the most influential mathematical economist of the early 19th century? A detailed analysis of this question is given in section 1.3 below. However, to complete the present argument, let us refer to the fact, as discussed in section 1.2.5, that Jevons and Walras were trying to design a *scientific economics* which was characterized mainly by its mathematical nature. This is exactly what Cournot had in mind. It is not surprising, therefore, that Jevons appraised Cournot without fully understanding him. It was the *compatibility* of Cournot's methodology and his attitude with those of Jevons and Walras which, after all, brought him recognition after 30 years. In summary, the higher levels of mathematical knowledge among economists in the 1870's as the main factor in understanding Cournot and thus reviving his work, do not count since Jevons himself did not possess such a mathematical knowledge.

1.2.3 Walras's Hypothesis: The Narrowness of Ideas

According to Walras,[8] the dichotomy between deduction and induction or between pure reasoning and experience which had separated sciences from arts was the main reason that classical economists disregarded the use of mathematics in their work. "If nineteenth century ... has completely ignored [mathematical economics], the fault lies in the idea, so bourgeois in its narrowness, of dividing education into two separate compartments: one turning out calculators with no knowledge whatsoever of sociology, philosophy, history, or economics; and the other cultivating men of letters devoid of any notion of mathematics" [*Elements of Pure Mathematics* (1900), 4th edition, p. 48]. Walras claimed that by employing both deductive and inductive reasoning, mathematical economics can be ranked with sciences such as astronomy and mechanics: "The twentieth century, which is not far off, will feel the need, even ... of entrusting the social sciences to men of general culture who are accustomed to thinking both inductively and deductively and who are familiar with reason as well as experience. The mathematical economics will rank with the mathematical sciences of astronomy and mechanics; and on that day justice will be done to our work" (*ibid*, p. 48).

Developments of mathematical economics in the 20th century have strongly rejected Walras's hypothesis. The question remains however that why Walras did not simply add the experimental dimension (i.e. quantitative analysis and measurements) to classical economics? In other words, if according to Walras, the *familiarity of economists with reason as well as experience* would have ranked economics with the acknowledged mathematical sciences, why, instead of completely ignoring the well-established classical economics, did he not make an effort to represent classical economics mathematically for the purpose of quantitative

[8]All references to Walras (1874), made in this chapter, are from its English translation by William Jaffé: *Elements of Pure Economics, or the theory of social wealth*, London: George Allen and Unwin, 1954, 620 pages.

analysis and empirical measurements?[9]

On the contrary, an examination of the *Elements of Pure Economics* indicates that Walras did not make any significant contribution either towards inductive thinking in economics or in measuring economic relations and advancing experimental aspects of economic theorization. His work, instead of taking classical economics one step towards experimentations, completely erased the empirical contents of classical political economy. In summary, Walras's actual contributions to economics did not follow his injunctions on the objectives of economic studies.

It is interesting to note that the academic life of Vilferedo Pareto confirms the contradiction existing between Walras's prime objective in economic theorization and his actual contributions. Recall that Pareto, accepted Walras's chair at Lausanne in 1892. After making a number of contributions mainly to Walras's theory of general equilibrium,[10] he realized the weakness of pure economics and its possible applied versions. In *Cours d' Économie Politique* (1896-7) he stated that "... pure economics shows us the general form of the phenomenon; applied economics provides a second approximations; but neither will even be able to show us how to manage the economic life of every individual" [Busino (1987), p. 801]. In *Cours* he clearly stated the importance of interrelations of economics and social phenomena. In 1905, Pareto published his *Manuale d'Economia Politica*; his words at the end of this book are clearly a departure from Walras's principles: "Whoever wants to make a scientific study of the social facts has to take account of reality and not of abstract principles and the like ...". Pareto then gave up

[9]For example, William Whewell, the Cambridge mathematician, represented mathematically some doctrines of political economy in general and Ricardo's system in particular. See Whewell (1829, 1831, 1850). His work were completely ignored by Walras.

[10]It should be noted that after accepting Walras's chair in 1892, "Pareto spent the whole of the next year writing a refutation of Marx's theory of value which was published in Paris in 1893 as the introduction to an anthology of passages by Paul Lafargue taken from Marx's *Das Kapital*". See Busino (1987), p. 801.

economics and concentrated exclusively on sociology. (*ibid*, p. 802).

In summary, with regard to the above facts, it is very difficult to accept Walras's hypothesis that the nineteenth century economists ignored mathematical economics simply due to the prevailing narrowness of ideas in discrediting experimentations in economic analysis. On the contrary, evidence is more in favour of the hypothesis that Walras's own contributions have further advanced and strengthened such narrowness of ideas.

1.2.4 von Neumann and Morgenstern's Hypothesis: The Unfavourable Circumstances

von Neumann and Morgenstern (1944) have examined the problem of mathematization of economics within a wider context. If economics is a science why, in contrary to other sciences where mathematics has been applied with great success, has its use not been highly successful? Most sciences could hardly make any progress without mathematics and yet the real contribution of mathematics to economics has remained an unsettled question. According to von Neumann and Morgenstern the combination of the following unfavourable circumstances are the main factors at work.

1. The Vagueness of Basic Economic Concepts. von Neumann and Morgenstern (1944, p. 4) have pointed out that "Economic problems were not formulated clearly and are often stated in such vague terms as to make mathematical treatment *a priori* appear hopeless because it is quite uncertain what the problems really are. There is no point in using exact methods where there is no clarity in the concepts and issues to which they are to be applied". This is in sharp contrast to the general view held among mathematical economists that the mathematization of economics facilitates a more concise exposition of problems and avoid the digressions of vague argumentation.

von Neumann and Morgenstern's claim implies the following contradiction: Further developments in mathematical economics as a *science* depends entirely on prior developments in "*non-scientific*" descriptive economics. To provide further evidence to our claim, we refer to page 4 (*ibid*): "Consequently, the initial task is to clarify the knowledge of the matter by further careful descriptive work". This is in dispute with the established view in the profession that if economics is to be a science it must be mathematical. Moreover, von Neumann and Morgenstern have not specified the conditions under which careful and concise advances in descriptive economics can be attained -i.e. with or without mathematics. If the latter holds, the uniqueness of mathematical economics as an exact science will collapse.

2. Inadequate Empirical Economic Facts. According to von Neumann and Morgenstern, mathematical economics has not achieved very much because the empirical background of economic science has been definitely inadequate. They held, however, that their comment should not be construed as a disparagement of statistical-economic research programme which was very promising at the time. (p. 5, *ibid*).

The recognition of the fact that statistical-economic research work, or what is now known as econometrics, could have made progress while mathematical economics was stagnating refers to the point that von Neumann and Morgenstern have admitted that economics as a science can make advances outside the restricted framework of Jevons's calculus of pleasure and pain or Walrasian utility and profit maximization.

3. Limitations in Mathematical Treatment of Human Behaviour. It appears that von Neumann and Morgenstern are the first mathematical economists of reputation in the twentieth century who have acknowledged the fundamental objection that economic theory cannot be modelled in the same format as physical sciences for it is a science of social and human phenomena which has to take into account a number of non-economic elements such as psychological, historical and cultural factors. This implies limitations in mathematical formulations of human behaviour: "We should attempt to utilise only some commonplace experience concerning human behaviour which lends itself to mathematical treatment" (*ibid.* p. 5).

von Neumann and Morgenstern have made a very important point that there are uncertainties about the exact mathematical methods which should be used in economic analysis in general and even in mathematization of that class of human behaviour which lends itself to mathematical treatment. The existing tools in mathematical economics such as calculus of variations or differential equations might not be the right instruments for economic analysis since they are mainly developed for physical sciences: "It is therefore to be expected -or feared- that mathematical discoveries of a stature comparable to that of calculus [11] will be needed in order to produce decisive success in [economics]. ... *A fortiori* it is unlikely that a mere repetition of the tricks which served us so well in physics will do for the social phenomena too" (*ibid.* p. 6).

In summary, von Neumann and Morgenstern's final recommendation to mathematical economists is to wait for new discoveries in mathematical methods which are more appropriate for the analysis of social sciences. But have they proved the existence of such mathematical methods? The answer is no. Moreover, before the discovery of such mathematical methods, how can the real economic problems "scientifically" be examined for policy recommendations? According to von Neumann and Morgenstern, economists cannot examine such real economic prob-

[11] In here, von Neumann and Morgenstern have referred to the role played by infinitesimal calculus in the creation of the discipline of mechanics.

lems simply because they are not yet qualified: "... How to stabilise employment, how to increase the national income, or how to distribute it adequately? Nobody can really answer these questions and we need not concern ourselves with the pretension that there can be scientific answers at present" (*ibid.* p. 6).

1.2.5 The Hypothesis of One-dimensionalisation of Economic Analysis: The Reduction of Economic Life to Mechanical Economic Science

We now propose a hypothesis to examine the three questions mentioned earlier. The growing desire for one-dimensionalisation of economic analysis in the second half of the nineteenth century strongly motivated the mathematization of economics. Let us start by a brief reference to the underlying factors which encouraged this approach in economic analysis.

Advances in classical economics ensued from contributions of Smith (1776), Ricardo (1817) and Mill (1848) together with further developments in theoretical socialism in general and Marxism in particular [Marx (1848, 1856-57 and 1867)], produced a state of multi-dimensionality in economic analysis in which economic issues were studied in relation to other dimensions such as historical, sociological and political aspects. There could be the following two inter-related responses to such an environment of multi-dimensionality in economic studies: "scientific" approach and "non-socio-political" i.e. "mathematical" approach.

1. Scientific and mathematical developments of the 18th and 19th centuries had a remarkable impact on many authors in shaping their attitudes towards searching for an economic science similar to physical or exact sciences. Close connections in all aspects of social life make a special study of any one of them not sufficiently productive. However, these authors favoured the idea that economists should assume their distinctive role and not devote themselves to study the laws

of a unified social science. The rapid progress in physical sciences in the 19th century was mainly regarded to be the result of breaking up the broad problems into their component parts. Such an outlook motivated some authors to accept the view that political economy should no longer be seen as if it were a single undivided and indivisible science.

The next step towards making a science of economics, in the same fashion as physical sciences, was to discover the general laws of economics which remain the same throughout all different ages and conditions: "Just as there is a general science of mechanics, so we must have a general science or theory of economy. ... The theory of economy proves to be, in fact, the mechanics of utility and self-interest" [Jevons (1876), pp. 198-199]. However, to some authors, economic science was identified with mathematical or *pure* economics which was confused on occasion with what is known today as econometrics. Since advances in mechanics had the greatest impact on the formation of mathematical economics, the role of calculus in mathematization of economics has always been profound.

2. Within an environment of conflicting theoretical developments between the mainstream classical economics and the newly established Marxian economics in the second-half of the 19th century, any significant theoretical contributions in classical economics usually had political and social implications, escalating the prevailing tension in political and social dimensions. Separation of economic issues from other related social subsystems could have been the remedy to this inter-related, multi-dimensional, "*chaotic*" state of economic studies. The concept of *pure* or *mathematical* economics, which was identical to economic *science*, was invented to represent a positive or politically neutral system of economic knowledge.

The Failure to Recognize Cournot?

Before starting to examine the role of calculus in one-dimensionalisation of economics, we are now in a position to provide an answer to our earlier question of how can the 30 years time lag in the recognition of Cournot (1838) be explained? Recall that according to Cournot (1838, p. 1), the wide varieties of different theories and doctrines had motivated the public desire for "positive" economics. However, despite Cournot's view, the social, political and economic conditions were not *chaotic enough* in Cournot's time to generate a strong driving force for one-dimensionalisation of economics. During the 30 years from Cournot to Jevons, the diffusion of socialism and Marxism had produced a different environment. When Jevons (1871) and Walras (1874) started their campaigns to popularise mathematical economics, the work of Cournot together with 38 work on mathematical economics before Cournot and 62 such work from Cournot to Jevons[12], were all revived to provide an army of supportive literature for the success of this new school of thought in economics. Even the antiquated work of Civa in 1711 was needed to give more strength to this army which was about to launch an attack upon the tradition of great classical economists as well as on Karl Marx's new unified and all-embracing approach to economic studies. We will come back to this point in section 2 below.

1. Mathematical Economics and the Formation of Mechanical Economic Science

Mathematical economics, in the sense discussed above, was developed in conjunction with the concept of *pure economics*. We will show in section 1.5 that mathematical economics cannot be defined properly without using the concept of pure economics and vice versa. This will play an important role in under-

[12]See Jevons (1871), pp. 322-339, for a complete list of these contributions.

standing the underlying factors at work in limitations of modern mathematical economics in general and economic applications of optimal control theory in particular. In this section, we first examine the nature of classical mathematical economics before presenting an analysis of mathematical economics as a remedy to multi-dimensional political economy.

i) The Nature of Classical Mathematical Economics

According to Jevons (1871, p. *vii* in preface), since economics "deals throughout with quantities, it must be a mathematical science in matter if not in language. ... The Theory of Economy thus treated presents a close analogy to the science of Statistical Mechanics and Laws of Exchange are found to resemble the Laws of Equilibrium of a lever as determined by the principle of virtual velocities. The nature of Wealth and Value is explained by the consideration of indefinitely small amounts of pleasure and pain, just as the theory of Statics is made to rest upon the equality of indefinitely small amounts of energy". Furthermore, on page 3 (*ibid*) he writes: "It is clear that economics, if it is to be a science at all, must be a mathematical science ... My theory of Economics, however, is purely mathematical in character".

By mathematics, Jevons basically meant differential calculus: "The theory consists in applying the differential calculus to the familiar notions of wealth, utility, value, demand, supply, capital, interest, labour and all the other quantitative notions belonging to the daily operations of industry" (p. 3). However, he held that the minimization of costs in fulfilling the utility of an individual is the ultimate objective of economic science which he defined as the *the Calculus of Pleasure and Pain* (1871, p. *vi*, preface). He wrote on page 27 that "the calculus of utility aims at studying the ordinary wants of man at the least cost of labour". Although Jevons has been usually praised for introducing into economic analysis the idea of maximization (or minimization), the origin of these ideas traces back to Cournot in 1838 when he wrote on page 44 in chapter 4 of his book that "we shall invoke but a single axiom, or, if you prefer, make but a single hypothesis, i.e. that each one seeks to derive the greatest possible value from his good or his labour".

From this point of view, the modern definition of economics as the allocation of scarce resources for optimum satisfaction of alternative objectives is in fact the generalization of Jevons's calculus of pleasure and pain; and modern mathematical optimization techniques in general and optimal control theory in particular, which are the most efficient mathematical tools for achieving such optimum satisfaction, are nothing but the advanced versions of elementary calculus employed by Jevons.

The nature of general mathematical methods in economic analysis has been remained basically unchanged since Jevons; although, as will be discussed in section 1.4, its role has been remarkably improved. According to Jevons (1879), "... the method consists in assuming certain simple conditions of the functions as comformable to experience and then disclosing by symbolic inference the implicit results of these conditions" (p. *xxxi* preface to the 2nd edition). As for the role of mathematics, we refer to Fisher (1891, p. 119) where he stated that "The efforts of the economist is to *see*, to picture the interplay of economic elements. ... Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. We see better. We also see further".

It is interesting to note that this idea can also be traced back to Cournot (1838). According to him the objective of using mathematical symbols is "to facilitate the exposition of problem, to render it more concise, to open the way to more extended developments and to avoid the digressions of vague argumentation" (p. 3). A simple comparison of Cournot's definition of the objective of mathematical economics with similar modern definition expressed by the editor of the *Journal of Mathematical Economics* reveals the fact that there has not been any significant change during the past 157 years. B. Allen (1995), the editor, expresses his view in the *Journal's* statement of aims as follows: "... the formal mathematical expression of economic ideas is of vital importance to economics. Such an expression can determine whether a loose economic intuition has a coherent logical meaning. Also a full formal development of economic ideas can itself suggest new economic concepts and intuitions. The primary objective is ... to express economic ideas using formal mathematical reasoning".

ii) Mathematical Economics: a Remedy to Multi-dimensional Political Economy

From the early 19th century, it was felt that the prevailing philosophical and historical approach to economic issues had produced a state of multi-dimensional political economy. Some economists, confused by such multi-dimensionalities and also impressed by rapid advances in pure and applied sciences, regarded this situation as the source of the prevailing alleged chaotic state of economic knowledge. Let us quote again from Cournot (1838) on this point: "The public is so tired of theories and systems that now the demand is for so-called 'positive' matters" (p. 1). A similar idea is expressed 40 years later by Jevons (1879): "The present chaotic state of Economics arises from the confusing together of several branches of knowledge" (p. *xvi* preface to the 2nd edition). Walras (1874) has also expressed similar views.[13]

[13] "There are today heaven knows how many schools of political economy: the *deductive* school and the *historical* school, the school of *laisser-faire* and the school of *State intervention* or *Socialism of the Chair*, the *Socialist* school properly so-called, the *Catholic* school, the *Protestant* school, etc. For my part, I recognize only two: the school of those who do not demonstrate and the school, which I hope to see founded, of those who do demonstrate their conclusions. By demonstrating rigorously first the elementary theorems of geometry and algebra and then the resulting theorems of the calculus and mechanics, in order to apply them to experimental data, we have achieved the marvels of modern industry. Let us follow the same procedure in economics and, without doubt, we shall eventually succeed in having the same control over the nature of

Subdivisionism was generally regarded to be a remedy to this alleged chaotic state of economics. For Cournot this subdivisionism meant constructing a pure or positive economic theorization which should be developed independent of the prevailing political systems: "I will only observe that *theory* ought not to be confounded with systems ... and that ... it should be permissible to consider from an exclusively theoretical standpoint, a subject of general interest which has so many different sides" (1838, pages 1 and 2 in the preface). Jevons (1871, p. 20) clearly signifies the necessity for subdividing economic knowledge: "Political economy is in a chaotic state at present, because there is need for subdividing a too extensive sphere of knowledge".[14] Moreover, "we must distinguish the empirical elements from the abstract theory, from the applied theory, and from the more detailed art of finance and administration. Thus will arise various sciences, such as commercial statistics, the mathematical theory of economics, systematic and descriptive economics, economic sociology and fiscal sciences ... Then will be division according to the manner of treating the branches of subject. The manner may be theoretical, empirical, historical, or practical; the subject may be capital and labour, currency, banking, taxation, land tenure, etc. -and not to speak of the more fundamental division of the science as it treats of consumption, production, exchange and distribution of wealth" (1879, p. *xvii*, 2nd edition).

Differentiating between scientific and literary temper in economic analysis can best be seen in Walras (1874) who named his book *Éléments d' Économie Politique Pure*. The fact that he was indebted to Cournot (1838) for using calculus in economic analysis implies that by pure economics he basically meant mathemat-

things in the economic and social order as we already have in the physical and industrial order" [Walras (1874), English translation (1954), p. 471].

[14]A similar idea is expressed by Jevons in a lecture on *the Future of Political Economy* delivered at the University College, London in 1878: "... One hundred years after the first publication of the *Wealth of Nation*, we find the state of the science to be almost chaotic. There is certainly less agreement now about what political economy is than there was thirty or fifty years ago" [see Jevons (1876), p. 191].

ical economics. On page 37 in the preface to the 4th edition of his book, Walras wrote as follows: "I readily acknowledge Gossen's priority[15] with respect to the utility curve and Jevons's priority with respect to the equation of maximum utility in exchange, but these economists were not the source of my ideas. I am indebted to my father, Auguste Walras, for the fundamental principle of my economic doctrine and to Augustin Cournot for the idea of using the calculus of functions in elaboration of this doctrine". Walras interchangeably used the terms mathematical economics and scientific economics to explain pure economics. This can clearly be seen in the introduction to the English translation of Walras's book *Elements of Pure Economics* in which William Jaffé, the translator, wrote in 1954 about how the economist Auguste Walras asked his son, Léon, to study mathematical economics at the age of 24 in order to build up a *scientific economics.*

Walras is perhaps the first mathematical economist who examined the problem of *abstraction* in pure or mathematical economics. Being impressed by advances in physical sciences, he argued that (1874, p. 71): "From real-type concepts, these sciences abstract ideal-type concepts which they define and then on the basis of these definitions they construct *a priori* the whole framework of their theorems and proofs. After that they go back to experience not to confirm but apply their conclusions". The surprising fact that Walras was not looking for any confirmation of the proposed mathematical model stems from his perception that "reality confirms these definitions and demonstrations only approximately and yet reality admits of a very wide and fruitful applications of these propositions" (*ibid*, p. 71). Following the same procedure, Walras defined the pure theory of economics

[15]In here Walras has referred to Hermann Heinrich Gossen who published his book *Entwickelung der Gesetze des Menschliches Verlehrs, und der daraus fliessnden Regeln fur menschliches Handeln* (The laws of human relations and the rules of human actions derived therefrom) in 1854. "[Gossen] remained an obscure civil servant all his life. His book, of which there is still a copy in the British Museum -the only one in existence possibly- was accidentally discovered by Professor Adamson and Stanley Jevons was again the first to recognize its merits", see Gide and Rist (1909, 1948), p. 491.

as a science which "ought to take over from experience certain type concepts, like those of exchange, supply, demand, market, capital, income, productive service and products. From these real-type concepts the pure science of economics should then abstract and define ideal-type concepts in terms of which it carries on its reasoning".

Despite the fact that Walras has not considered the crucial issue of the *method of abstracting* ideal-type concepts from real-type concepts, the practice of modern econometrics invalidates his hypothesis that testing mathematical models against reality cannot be possible due to the existence of inherent approximations. However, when Walras was faced with the question that "do these pure truths find frequent applications?" he responded with the following set of contradictory answers.

1) Walras goes to one extreme by saying that it is not the aim of pure economics to provide solutions to real-life economic problems; it only furnishes an academic pleasure to an economic scholar: "To be sure, a scholar has a right to pursue science for its own sake, just as the geometer has the right (which, in fact, he exercises every day) to study the most singular properties of geometrical figures, however fantastic, if he finds that they excite his curiosity" (*ibid*, p. 71-72).

2) Walras goes to the other extreme by saying that pure economics cannot only solve real economic problems but can control the nature of things in economics and social order exactly in the same manner as physical and industrial order are in control: "By demonstrating rigorously first the elementary theorems of geometry and algebra and then the resulting theorems of the calculus and mechanics, in order to apply them to experimental data, we have achieved the marvels of modern industry. Let us follow the same procedure in economics and, without doubt, we shall eventually succeed in having the same control over the nature of things in

economics and social order as we already have in the physical and industrial order" (*ibid*, p. 471).

3) Walras has also taken a middle-way. He has reduced the difficulties associated with applications of mathematical economics simply to a set of technical complications which can easily be treated by other economists in due course: "... practically all the criticisms levelled against me have consisted in calling my attention to complications which I had left to one side. I find it very easy to reply to these criticisms. So far as I am concerned, since I was the first to elaborate a pure theory of economics in mathematical form, my aim has been to describe and explain the mechanism[s] ... in terms of [their] bare essentials. It is for other economists who come after me to introduce one at a time whatever complications they please. They in their way and I in mine will then, I think, have done what had to be done" (*ibid*, p. 478). Unfortunately, Walras has ignored the very important problem that how the "ideal-type" economic concepts can embrace the increasing number of complexities existing in real-economic life and at the same time maintain its abstract nature which is so essential for mathematization of economic behaviour.[16] This is the key question to the limitations of mathematical economics which will be briefly discussed in section 1.4.

[16]Like Walras, Charles Roos, a founder of the *Econometric Society* in 1930, (with Ragnar Frisch and Irving Fisher), has confused the structural shortcomings of mathematical economics (due to the abstraction method discussed in section 1.6) with the number of explanatory variables in a behavioural equation. In an article published in the second volume of *Econometrica* (no. 1, January 1934), he stated that "So many mathematical economists -Cournot, Walras, Pareto, Fisher, Frisch, Evans, Schultz and others- have already given such excellent reasons for employing mathematics in economics that it seems unnecessary for me to add anything. However, ... some economists and others have said that there are so many variables involved in a study of human behaviour that it will never be possible to develop a science of economics. To these who would use these as an argument for not using mathematics in economics, one might reply that because there are so many variables, there is all the more need for an exact language to keep track of them" pp. 73-74.

2. Mathematical Economics and Marxian Economics

The emergence of mathematical economics as a discipline has coincided chronologically with the diffusion of Marxian economics.[17] This proffers the hypothesis that the revival of mathematical economics in the 1870s could have possibly been a response to dissemination of socialism in general and Marxian economics in particular.

Marxian economics is an integrated body of knowledge which aims at studying economic issues within a three dimensional sphere of philosophical (German philosophy), historical (dialectical materialism) and classical economics (Smithsonian and Ricardian schools). In the Marxian tradition, every fundamental economic concept, such as value or capital, can best be explained within this three dimensional space. By concentrating only on the economic dimension and ignoring the feedback mechanism with philosophical and historical aspects, pure or mathematical economics necessarily depleted the real content of fundamental economic concepts. This naturally downgraded the aim of economic analysis to a simple logical or mathematical inference based on some basic assumptions which were laid upon such empty concepts. An examination of the concept of value might reveal this fact. The concept of value in Marxian economics with its class dimension, was reduced to the concept of return, pleasure, or utility in mathematical economics. In this approach, class had been replaced by an individual economic agent, thus the historical dimension associated with class developments (dialectical materialism) and its relation to value (the formation of surplus value and exploitation) will become effectively futile or irrelevant within the context of pure or mathematical economics.

[17]Marx's contributions before Jevons (1871) included the followings: *Manifesto of the Communist Party* (1848), *Contribution to the Critique of Political Economy* (1858), *Grundrisse: Foundations of a Critique of Political Economy*, (written in 1858-59, first published in 1939) and *Capital*, vol. 1, 1867.

As an example, consider Jevons (1871) or Walras (1874) who defined utility as the origin of value: "Repeated reflection on inquiry have led me to the somewhat novel opinion, that *value depends entirely upon utility.* Prevailing opinions make labour rather than utility the origin of value; and there are even those who distinctly assert that labour is the *cause* of value ... Labour is found often to determine value, but only in an indirect manner, by varying the degree of utility of the commodity through an increase or limitation of the supply" [Jevons (1871), p. 1]. On the basis of a one-dimensional concept of utility, Jevons (1871) defined his aim as to find the laws of variations of utility and to derive a theory of exchange on the basis of utility: "We have only to trace out carefully the natural laws of the variations of utility, as depending upon the quantity of commodity in our possession, in order to arrive at a satisfactory theory of exchange, of which the ordinary laws of supply and demand are a necessary consequence" (p. 1). One-dimensionalisation of value was further advanced when Jevons in his *Principle of Economics*, which was published after his death in 1905, stated that "... as value, after all, is but a development of utility, I have seen reason to take utility rather than value as the subject-matter of economics" (p. 49).

The implications for historical and philosophical aspects of economic analysis are interesting. The community of mathematical economists believed that these topics would be the subject matter of the science of the evolution of social relations, i.e. the newly established science of sociology. According to Jevons (1871, p. 20), "instead of converting our present science of economics into an historical science, utterly destroying it in the process, I would perfect and develop what we already possess and at the same time erect a new branch of social science on an historical foundation. This new branch of science ... is doubtless a portion of what Herbert Spencer calls Sociology, the Science of the Evolution of Social Relations".

The above argument does not however constitute a solid basis on which one can completely accept the hypothesis that the revival of mathematical economics in the 1870's has been a response to Marxism; because without Marxism one should not have expected the developments in mathematical economics- a hypothesis which is difficult to prove. However, we cannot disregard the fact that opponents to Marxism would have been very pleased with the emergence of mathematical economics, for instead of challenging Marxian economics here and there the whole body of Marxian economics could have been safely left to one side. The subdivisionism, propagated so strongly by Jevons (1871) and Walras (1874) to justify the pure or mathematical economics as an abstract science, was in a sharp contrast to the underlying Marxian notion of *economic life* within the "science of society". This Marxian approach strongly challenges the existence of an economic science which is isolated from sociology, anthropology, politics, etc. Moreover, the basic underlying assumptions in mathematical economics, i.e. an individual economic agent and the utility maximization principle, together with the dynamics of mathematical reasoning in which implicit results are inferred from the assumed conditions, made the machinery of Marxian economics irrelevant in the new discipline of mathematical economics. This explains the fact that despite having topics on value, Jevons (1871) as well as Walras (1874) could have managed their arguments on this subject without making even one single reference to Marx or his work on value.[18]

Whether the formation of mathematical economics in the 1870's has been the outcome of attempts to establish an economic science similar to physical sciences or has been a response to Marxism- or a combination of both- the fact is that

[18]Jevons did not have any chapter on value in his *Theory of Political Economy* (1871). However, chapter *vii* in his *Principles of Economics* (1905) is on value. Chapter 16 (or Lesson 16) in Walras's *Elements of Pure Economics* is on the "Exposition and refutation of Smith's and Say's doctrines of the origin of value in exchange". Neither in this chapter, nor anywhere else in his book, has Walras made any reference to Marx or the Marxian theory of value.

it has transformed the studies of real economic life to an abstract mechanical science of economics. An important point is that the real driving force behind such transformation has neither been a theoretical motive nor an empirical necessity; otherwise the eminent classical economists would have taken the initiative in the formation of mathematical economics. No single acknowledged economist has played any significant role in setting up mathematical economics. Cournot was a mathematician; Jevons studied mathematics, logic and chemistry; Walras studied mathematics and engineering; and Pareto was an engineer.[19] As will be discussed in section 1.3, there was a period of 30 years after Marshall (1890) when no significant contributions in mathematical economics were observed; the community of economists had apparently remained faithful to their old tradition of political economy. Again, it was the mathematicians who revived mathematical economics. Ramsey (1928), Kantorovich (1939), and von Neumann (1928 and 1944) broke the silence and made the initial individual moves towards the modern era of mathematical economics; but this time within a different framework which will be discussed in section 1.3.2.

[19]It is interesting to note that these authors used all possible means to convince classical economists that the new discipline of mathematical economics was highly important. In his introductory lecture at the opening session 1876-1877 at University College, London, Jevons warned British economists strongly in the following words: "It may be safely asserted, however, that if English economists persist in rejecting the mathematical view of their science, they will fall behind their European contemporaries. How many English students, or even Professors, I should like to know, have sought out the papers of the late Dr. Whewell, printed in the *Cambridge Philosophical Transactions*, in which he gives his view of the mode of applying mathematics to our science? What English publisher, I may ask again, would for a moment entertain the idea of reprinting a series of mathematical work on political economy? Yet this is what is being done in Italy by Professor Gerolamo Baccardo, the very learned and distinguished editor of the *Nuova Enciclopedia Italiano* ... Now, too, that attention is at last being given to the mathematical character of the science, it is becoming apparent that a series of writers in France, Germany, Italy and England have made attempts towards a mathematical theory. Their work have been almost unnoticed, or, at any rate, forgotten, mainly on account of the prejudice against the line of inquiry they adopted ... On the present occasion, I cannot do more than mention the names of some of the principal writers referred to, such as Lang, Kroeneke, Buquoy, Dupuit, von Thunen, Cazaux, Cournot and Francesco Fuoco, on the continent; and Whewell, Tozer, Lardner, Peronnet Thompson, Fleeming Jenkin, Alfred Marshall and probably others, in Great Britain", Jevons (1876), pp. 199-200.

1.3 The Origin and Formation of Modern Mathematical Economics

It is generally agreed [for example, Debreu (1986)], that the symbolic beginning of contemporary mathematical economics is 1944, when the well-known mathematician John von Neumann and the economist Oskar Morgenstern published their work on *the Theory of Games and Economic Behaviour.* This book "sets a new level of logical rigour for economic reasoning" [Debreu (1986), p. 1261] and presented for the first time a new mathematical method for economic analysis, i.e. the game theoretic approach. However, to examine the validity of Debreu's hypothesis we refer briefly to the developments in mathematical economics which have taken place in the interval between Marshall and von Neumann.

While being very conservative in the use of mathematics in economic analysis, Marshall's *Principles of Economics* (1890) was a synthesis of classical economics of Smith, Ricardo and Mill and classical mathematical economics developed by Cournot, Jevons and Walras (see section 1.4.2). Marshall, who was a mathematician before becoming an economist, gave special emphasis to non-mathematical analysis and kept all his mathematical presentations in a long appendix to his book. Even most diagrammatic representations were included in footnotes. His book dominated economic literature for more than 30 years.[20] It can be claimed that Marshall's book was one of the main sources of economic knowledge in capitalist economies before the Keynesian revolution in 1936.

It seems that during the period from Pareto (1897) and Marshall (1890) to the 1930's, one cannot identify a course of continual developments in mathematical economics. Different individual contributions can be classified either as attempts to reorganise and consolidate previous mathematical treatments of economics or

[20]The 8th edition of the *Principles of Economics* was published in 1920 in 850 pages. The 9th edition, published in 1961, can be regarded as a reprint of a classical work.

to provide new and comprehensive reports on the significance of already known results. Bowley, Evans and Wicksell are the three most important writers on mathematical economics in this period.

Bowely published his *Mathematical Groundwork of Economics* in 1924. Despite being an acknowledged and well-respected statistician and economist at the London School of Economics, his work did not have any impact on the direction of research work in economics. Evans, who studied mathematics at Harvard and was a professor of mathematics at Berkeley in 1934, published his *Mathematical Introduction to Economics* in 1930. This, together with his earlier contributions on the applications of calculus of variations to economic analysis [Evans (1925)] did not influence the prevailing state of economic research. Wicksell, who intended to become a professor of mathematics but studied economics upon completing his doctorate in mathematics, published his *Lectures on Political Economy* in 1934, again with the same result as Evans's and Bowley's contributions.

The highly inventive use of calculus of variations in economic analysis in the 1920's has been the most important contribution of mathematical economics which did not significantly affect the profession at the time. Frank Ramsey, a well-known young Cambridge mathematician, successfully applied this method to study saving behaviour. Ramsey contributed two papers to the literature of mathematical economics. His first paper,appeared in 1927 in the *Economic Journal*, was on the theory of taxation. However, it was his second paper, published in 1928, again in the *Economic Journal*, which is regarded as an epoch-making contribution to economic optimization. Keynes (1972, pp. 335-336) stated that Ramsey's second paper "is, I think, one of the most remarkable contributions to mathematical economics ever made ...".

Unfortunately, as Koopmans (1965) reports, Ramsey's contribution was almost totally ignored by economists until the middle 1960's. We will come back to this point in section 1.6 where economic applications of optimal control theory are studied. It should be noted however that historically, the earliest attempts to apply variational methods to economic analysis can be extended back to Edgeworth (1881). Besides, the work of Hotelling (1925), Evans (1925) and Roos (1928) are known to be good examples of such applications. Before closing our discussion on the theoretical contributions on mathematical economics in the 1920's, let us add the von Neumann's contribution on game theory and its potential applications to economic analysis in 1928 to the above list. This work, which was published in German, was also totally ignored by economists until von Neumann and Morgenstern published their book on *the Theory of Games and Economic Behaviour* in 1944.

Despite all these individual contributions, the community of economists, quite sensibly, had remained faithful to their non-mathematical Marshallian type economic analysis. Under such circumstances, it is difficult to accept Debreu's hypothesis that von Neumann and Morgenstern's *Theory of Game and Economic Behaviour* (1944) has marked the beginning of a new era in mathematical economics. In what follows, we first examine the evidence which contradicts Debreu's hypothesis.

1.3.1 Refutation of Debreu's Hypothesis on von Neumann and Morgenstern's Epoch-making Contribution

According to the following evidence, we find that Debreu's hypothesis, which regards the publication of *Game Theory and Economic Behaviour* in 1944 as the symbolic birth of modern mathematical economics, is not convincing.

1) Earlier contributions were equally important. Recall that Keynes (1972, pp. 335-336) regarded Ramsey's second paper (1928) on optimal levels of saving

as "one of the most remarkable contributions to mathematical economics ever made". Moreover, contributions of Leonid Kantorovich (1939) on organizing and planning of production and Wassily Leontief (1941) on input-output analysis were so significant that they were awarded the Nobel prizes in 1975 and 1973, respectively.

2) Recall that von Neumann and Morgenstern have made the strongest attack on mathematical methods used in solving consumer's utility maximization and producers profit maximization in Walrasian type mathematical economics. They claimed that the exact description of an economic agent's effort to attain maximum satisfaction can be obtained by employing a game-theoretic approach: "It is well known that considerable -and in fact unsurmounted- difficulties this task [utility or profit maximization] involves given even a limited number of typical situations ... It will appear, therefore, that their exact positing and subsequent solution can only be achieved with the aid of mathematical methods which diverge considerably from the techniques applied by older or by contemporary mathematical economics ... [i.e.] the mathematical theory of *games of strategy*" (1944, p. 1).

The publication of *Theory of Games and Economic Behaviour* in 1944 did not strongly influence the direction of research in mathematical economics. The main contribution of this work has been the application of mathematical theory of "games of strategy" developed by von Neumann in 1928 to the analysis of economic behaviour. Moreover, they have tried to make it clear that their objective is not to produce an analogy between economic behaviour and game theory but to establish that "the typical problems of economic behaviour become strictly identical with the mathematical notions of suitable games of strategy" [von Neumann and Morgenstern (1944), p. 2]. Their prime goal was, therefore, to introduce a new mathematical tool, i.e. *game theory* for the analysis of the well-known fundamental economic questions. We can see this fact clearly from the three opening lines of their book: "The purpose of this book is to present a discussion of some fundamental questions of economic theory which requires a treatment different from that which they have found thus far in the literature".

From a purely mathematical point of view, von Neumann and Morgenstern's work is a departure from Walrasian tradition and hence can be regarded as the beginning of a new era in mathematical economics. But their work was overlooked by the community of mathematical economists until recently when the game-theoretic approach has been increasingly applied in microeconomics. In summary, their contribution did not change the course of developments in mathematical economics following 1944 and thus cannot be valued as the beginning of a new phase in mathematical economics.

3. Despite von Neumann and Morgenstern's emphasis on a game-theoretic approach in economic analysis, the actual developments of mathematical economics after 1944 were mainly along the analysis of general equilibrium and optimal properties of growth models in the 1950's and 1960's. Although von Neumann's generalization of Brouwer's fixed point theorem to prove the existence of an optimal growth path has been a remarkable contribution, this has no relation with the game-theoretic approach to economic behaviour.

One cannot ignore that von Neumann and Morgenstern's contribution provided a profound and extensive new level of mathematical rigour in economic reasoning. The introduction of *convex analysis* into economic theory (1944, chapter III, section 16) has made the concept of *convexity* an integral part of the main topics in mathematical economics such as consumption theory, production theory, welfare economics, efficiency analysis and more importantly in the theory of general equilibrium. Again, these minor contributions cannot count as the beginning of a new era in mathematical economics.

1.3.2 The Hypothesis of Coordinated Research Programmes

A new phenomenon which has been observed in the beginning of 1930's is the formation of academic research institutions to coordinate and encourage advances in mathematical economics. Let us mention at the outset that there has been no coordination in research work in the first phase of developments in mathematical economics, i.e. during the 1870's. Despite independent developments of Jevons's and Walras's contributions, Walras initially respected Jevons's work and acknowledged his priority in formulating the *Equation of Exchange* which was identical to Walras's *Condition of Maximum Satisfaction.* However, soon Walras took an unfriendly position against Jevons and regarded him as a plagiarist of his work. In fact, there was not any research coordination between Jevons, Walras and Pareto. In section 1.2.3 I observed how Pareto, the successor to Walras in the chair of political economy at Lausanne, departed from Walras's tradition and gave up economics and concentrated exclusively on sociology.[21]

[21] To provide evidences on the absence of coordination in the classical mathematical economics, we refer to Walras (1874). In the preface to the first edition (pages 35 and 36) he stated as follows: "This work was completely written and almost completely printed...when...my attention was drawn to a work on the same subject, entitled: *The Theory of Political Economy*, in 1871, by W. Stanley Jevons, Professor of Political Economy at Manchester...I acknowledge Mr. Jevons's priority so far as his formula is concerned, without relinquishing my right to claim originality for certain important deductions of my own. I should not enumerate these points which competent readers will readily discover. I need only add that, as I see it, Mr. Jevons's work and my own, far from being mutually competitive in any harmful sense, really support, complete, and reinforce each other to a singular degree".

The above passage might indicate a harmony between the work of Jevons and Walras; but the lack of coordinated research programme, soon put an end to this friendly attitude. Donald Walker (1987, pp. 861-862) reports that "[Walras's] initial cordiality towards Jevons, as a fellow pioneer in mathematical economics, was dissipated by Jevons's failure to recognize Walras's [main] contributions ... and eventually Walras ... came to regard Jevons as a plagiarist of his work (Walras to M. Pantaleoni, 17 August 1889) ... Walras felt neglected by Alfred Marshall [too] ... Walras wrote in 1904 that "I have not the least doubt about the future of my method and even my doctrine; but I know that success of this sort does not become clearly apparent until after the death of the author" (Walras to G. and L. Renard, 4 June 1904)".

The starting point in the formation of modern mathematical economics has been the realization of the fact that the revival of classical mathematical economics necessarily required an integration of statistical techniques into mathematical economic analysis. The creation of the *Econometric Society* on December 29, 1930, in Cleveland, Ohio, was mainly for the unification of *economic theory*, *mathematical analysis* and *statistics*. The early attempts by Ragnar Frisch, Professor of Economics, University of Norway, Oslo; Charles Roos, Permanent Secretary of the American Association for the Advancement of Science, Washington; and Irving Fisher, Professor of Economics, Yale University, had a profound significance on the creation of Econometric Society. Irving Fisher became the President and Chairman of the Council, while the other two, together with Arthur Bowley, Professor of Statistics at LSE, Joseph Schumpeter, Professor of Economics, Harvard University and Alfred Cowles,[22] Director of the Cowles Commission for Research in Economics,[23] (as treasurer), formed the Council of 10 members. Section *I* of the Constitution reads as follows: "The Econometric Society is an international society for the advancement of economic theory in its relation to statistics and mathematics".

Econometrica, the journal of the Econometric Society, began publication in January 1933. Ragnar Frisch was appointed editor. It is interesting to note that, in line with the Constitution, the associate editors represented the emphasis on economics, mathematics and statistics: Alvin Hansen, Professor of Eco-

[22]It should be noted that Alfred Cowles was not among the first Council of 10 members in the *Econometric Society*. When L. V. Bortkiewicz from University of Berlin, who was a member of Council, died in August 1931, Alfred Cowles was appointed as a member and the treasurer of the Council.

[23]The Cowles Commission for Research in Economics, founded in 1932 by Alfred Cowles and a group of economists and mathematicians concerned with the applications of quantitative techniques to economics and the related social sciences at Colorado Springs. The Commission moved to Chicago in 1939 and was affiliated with the University of Chicago until 1955 when it moved to Yale. The research staff of the Commission along with other members of the Yale Department of Economics established the Cowles Foundation for Research in Economics to sponsor and encourage the development and application of quantitative methods in economics and related social sciences.

nomics from University of Minnesota, Frederick Mills, Professor of Statistics from Columbia University and Harold Davis, Associate Professor of Mathematics from Indiana University.

To clarify the importance of the Econometric Society in advancing the new phase of mathematical economics it is opportune to refer to some points which Ragnar Frisch made in his first editorial to Econometrica. "Econometrics is by no means the same as economic statistics. Nor it is identical to what we call general economic theory, although a considerable portion of this theory has a definitely quantitative character. Nor should econometrics be taken as synonymous with the application of mathematics to economics. Experience has shown that each of these three view points, that of statistics, economic theory and mathematics, is a necessary, but not by itself a sufficient, condition for a real understanding of the quantitative relations in modern economic life. It is the *unification* of all three that is powerful. And it is this unification that constitutes econometrics" (p. 2).

According to Ragnar Frisch in his first editorial (*ibid*), the mutual penetration of quantitative economic theory and statistical observation, which is the essence of econometrics, can profoundly change the tradition of economic theorization in classical mathematical economics: "Theory, in formulating its abstract quantitative notions, must be inspired to a large extent by the technique of observation. And fresh statistical or other factual studies must be healthy elements of disturbance that constantly threatens and disquiet the theorist and prevents him from coming to rest on some inherited, obsolete set of assumptions".

The emphasis given by Ragnar Frisch on the role of mathematics in quantitative economics is interestingly confusing. On the one hand, he maintained that (*ibid*, p. 2) "Mathematics is certainly not a magic procedure which in itself can solve the riddles of modern economic life, as is believed by some enthusiasts. But when combined with a thorough understanding of the economic significance of the phenomena, it is an extremely helpful *tool*". On the other hand, he held the view that "Indeed, it will be an editorial principle of Econometrica that no paper shall be rejected solely on the ground of being too mathematical. This applies no matter how highly involved the mathematical apparatus may be" (pp. 2-3).

The coordinated programme of research work mapped out by the Econometric Society[24] was not mutually competitive in any harmful sense with traditional classical mathematical economics of Jevons and Walras; on the contrary, it really completed it. To respect the founder of classical mathematical economics, the first volume of Econometrica is opened by a portrait of Cournot accompanied by a paper in French entitled "Cournot et L'École Mathematique" by René Roy.

Applications of statistical methods to economic analysis, advocated so strongly by the Econometric Society, produced a more positive attitude towards measurement in economic analysis. Recall that utility maximization was the corner-stone of the contributions of Jevons, Walras and Pareto. It is not surprising that a debate concerning *determinateness of utility* was given higher priority among the first serious coordinated research work. In this debate, questions of *cardinality* and *ordinality* in utility functions were discussed in detail. The work of Lange (1934, 1935), Phelps-Brown (1934), Allen (1935) and Bernandelli (1935), in the new journal of *the Review of Economic Studies*, (first published in 1933), were among the most significant contributions. The Econometrica and the Review of

[24]Contributions of Cowles Commission for Research in Economics towards further developments of such coordinated research programmes should not be overlooked. As mentioned earlier, this Commission was established in Colorado in 1932, moved to Chicago in 1939 and to Yale (as Cowles Foundation) in 1955. Its main contributions in advancing quantitative methods in economics are summarised in the *Report of Research Activities* as follows: "The activity analysis formulation of production and its relationship to the expanding body of techniques in linear programming became a major focus of research at Chicago period. The Walrasian model of competitive behaviour was examined with a new generality and precision, in the midst of an increased concern with the study of interdependent economic units and in the context of a modern reformulation of welfare theory" [see Cowles Foundation (1983), p. 1].

Economic Studies, were acting as continuous sources of encouragement for further research work in mathematical economics.

Contributions of Leonid Kantorovich (1939) on organizing and planning of production, Wassily Leontief (1941) on input-output analysis, Paul Samuelson (1947) on foundations of economic analysis, Tjalling Koopmans (1951) on activity analysis of production and George Dantzig (1951) on simplex algorithm are examples of the results obtained in the new optimistic early period of modern mathematical economics. Further expansion in the literature encouraged the publication of more specialised journals in mathematical economics: *International Economic Review* in 1960, *Journal of Economic Theory* in 1969, *Journal of Mathematical Economics* in 1974 and *Journal of Economic Dynamics and Control* in February 1979 are the well-known examples which have stimulated further research interests in the subject.

A new generation of economists, therefore, emerged who were equipped with greater mathematical knowledge and whose attitudes and tools of analysis were more statistical and mathematical as compared with their predecessors. I consider the above mentioned academic environment together with the attitude and character of the new generation of economists, which have produced, in turn, a growth generating cycle in mathematical economics, as the origin of the new phase in mathematical economics. We will discuss more on this point in section 1.4.2 where mathematics as an engine of inquiry in modern mathematical economics is discussed.

Having established the importance of coordinated research programmes and appropriate academic institutions in the initiation and advancement of modern mathematical economics, let us briefly examine the very important topics of attitudes and limitations in both classical and modern mathematical economics.

1.4 Classical vs Modern Mathematical Economics: Attitudes and Limitations

What are the salient features of contemporary mathematical economics which have made it distinctive from classical or traditional mathematical economics? An answer to this question might provide an understanding of the future directions in this subject; this, in turn, will partially identify the status of economic applications of optimal control theory in mathematical economics. Historical analysis of mathematical economics reveals that the attitudes of mathematical and non-mathematical economists towards the subject as well as their perceptions on the limitations of mathematical methods in economic analysis are the two important factors which have differentiated the old and new schools of mathematical economics.

1.4.1 Attitudes

Recall that we have classified the pre-1930's literature on mathematical economics as *classical* and the work done since the creation of the Econometric Society to the present time as *modern* mathematical economics. Classical mathematical economists had, more or less, an inimical attitude towards non-mathematical economists. This feature is basically absent among contemporary mathematical economists because the number of those economists who do not believe in mathematization of economics are either very limited or their influences in directing major coordinated economic research programmes are marginal.

Let us start by considering the attitudes of classical mathematical economists. Cournot (1838, p. 2 in the preface) held the view that there exists a strong prejudice among writers in political economy against mathematical economics: "But the title of this work [*Researches into the Mathematical Principles of the Theory of Wealth*] ... shows ... that I intend to apply to them the forms and symbols of mathematical analysis. This is a plan likely, I confess, to draw on me at the outset the condemnation of theorists of repute. With one accord they have set themselves against the use of mathematical forms and it will doubtless be difficult to overcome to-day a prejudice which thinkers, like Smith and other more modern writers, have contributed to strengthen".

Classical non-mathematical economists on the other hand generally believed that an economist should be capable of explaining factual economic issues and should have a clear understanding of different economic doctrines. Under such circumstances, it is quite reasonable that classical writers in political economy showed no sympathy with Cournot's mathematical economics when Cournot himself rightly believed (1838, p. 3) that there was a wide gap between the existing economic knowledge and practical issues in real economic life: "I believe that there is an immense step in passing from theory to governmental applications; ... and I believe, if this essay is of any practical value, it will be chiefly in making clear how far we are from being able to solve, with full knowledge of the case, a multitude of questions which are boldly decided every day". However, Cournot did not demonstrate how mathematical economists could possibly bridge this gap.

Jevons (1879, p. *xxiii*, 2nd edition) took a different unfriendly position against non-mathematical classical economists. As a convinced mathematical economist he argued that " ... economists have long been mathematicians without being aware of the fact. The unfortunate result is that they have generally been bad mathematicians and their works must fall". His opposition to non-mathematical approach had developed to the extent that he ruled out the possibility of any contributions from non-mathematical school to his so-called pure or scientific economics: "Instead of converting our present science of economics into an historical science, utterly destroying it in the process, I would perfect and develop what we already possess" (1879, p. 20). On the other hand, the stand which non-mathematical classical economists took on the new emerging science of economics was more logical and confined to the analysis of the usefulness and relevance of mathematical economics. As will be discussed below and also in section 1.6, these two strong observations are still valid.

To the extent that mathematical methods are defined as the instruments of drawing necessary conclusions from given assumptions and conditions, there seemed to be no dispute on utilisation of mathematical methods in economics where such applications were possible. Moreover, it was agreed that the utilisation of mathematical symbols, operations and geometrical representations in economic reasoning would not only facilitate the exposition and generalization of problems but also rendered them to greater precision of statements by avoiding vague argumentation. Conflicting views arose upon the proposition that whether the application of mathematics to economics could produce *economic results* which were previously *unknown.* For example, Jevons (1879), in appraising Cournot's book wrote "... this investigation, presents a beautiful example of mathematical reasoning, in which knowledge is apparently evolved out of ignorance" (p. *xxxi*, preface to the 2nd edition).

Perhaps the strongest objection by classical economists to this line of argument has been put forward by Cairnes (1875, preface): "Having weighted Professor Jevons's argument to the best of my ability and so far as this is possible for one unversed in mathematics, I still adhere to my original view. So far as I can see, economic truths are not discoverable through the instrumentality of mathematics. If this view be unsound, there is at hand an easy means of refutation-the production of an economic truth, not before known, which has been thus arrived at; but I am not aware that up to the present any such evidence has been furnished of the efficiency of the mathematical method".

Despite some elements of truth in Cairnes argument, advances in modern mathematical economics imply that the applications of mathematical methods to economic analysis can provide some real economic contributions. Examples are linear programming in resource allocations and input-output models in planning. This point will be further examined in section 1.4.2.

The argument about whether mathematical economics can play any crucial role in realization of the primary objectives of economics is a strong criticism against mathematization of economics. This objection has been put forward initially by Wagner (1891, p. 327): "I do not believe that this mode of treating the subject has an independent value of its own for solving our problems". It is interesting to examine what were "our" problems. The ultimate objective of classical economy was to explain the economic behaviour as observed in real economic life. Under such circumstances, it is not surprising that the classical mathematical economics, which was nothing more than the calculus of pain and pleasure [Jevons (1871, preface)] did not receive a warm welcome by classical economists.

However, an important question is the relevance of modern mathematical economics in solving contemporary economic problems. The answer to this question depends partly upon the usefulness of modern mathematical economics and partly on the *character of contemporary economists.* We will discuss the former in section 1.7, but let us now examine the latter.

The average or expected attitude of economists changes from one generation to the other according to what is in fashion at the time. Young economists do not consciously choose their mode of economic training; they are simply born into a predetermined world of educational economic institutions and can only shape their education by selecting from a given list of economic subjects.

This explains why those who have established the foundations of mathematical economics were not formally trained within the tradition of classical economics: Cournot (1838), a mathematician; Jevons (1871), basically trained in mathematics, logic and chemistry; Walras (1874), an engineer but self-taught in economics; Pareto (1896), an engineer; and Ramsey (1928) and von Neumann (1944) leading mathematicians. It took nearly two to three generations from Jevons for the general attitudes of economists to move gradually away from classical political economy and become closer to the mathematical approach in economic training. This was well predicted by Fisher when he wrote in 1891 (p. 119) that "It may not be rash to expect that the next generation of the theoretical (as distinct from historical) economists will have fitted themselves by mathematical training for this mode of treating their theme and that they will be such men as by natural aptitude can so fit themselves". It follows that contemporary economists can challenge the relevance and usefulness of one or two subjects in economics but cannot usually question the validity or relevance of the whole existing system of mathematical economics because there does not exist an alternative system of acknowledged and academically approved economic knowledge.

1.4.2 Limitations

Classical mathematical economists generally accepted the existence of limitations in applying mathematical methods to economic analysis. Modern mathematical economists do not usually share this reservation. Let us first briefly examine the perceptions of classical mathematical economist on such limitations. It seems that they basically approved only those mathematical applications which led to results of value. Jevons (1879, p. *xxiii*) explains this very clearly: "It does not follow, of course, that to be explicitly mathematical is to ensure the attainment of truth and in such writing as those of Canard and Whewell, we find plenty of symbols and equations with no result of value, owing to the fact that they simply translated into symbols the doctrines obtained, and erroneously obtained, without their use".[25]

Despite the ambiguity of the term "result of value", one can argue that the use of symbols and mathematical operations in economic analysis can be justified on the ground that they lead to a better exposition of economic issues which is itself a result of value. This view had been first expressed by Cournot (1838, p. 3): "The employment of mathematical symbols is perfectly natural when the relations between magnitudes are under discussion; and even if they are not rigorously necessary, it would hardly be reasonable to reject them, because they are not equally familiar to all readers and because they have sometimes been wrongly used, if they are able to facilitate the exposition of problems". However, unlike Jevons (1871, p. 3) who believed that political economy "must be mathematical simply because it deals with quantities", Cournot (1838, p. 5) allowed for topics in political economy which cannot be treated mathematically: "I have not set out to make a complete and dogmatic treatise on Political Economy; I have put aside questions, to which mathematical analysis cannot apply".

Unfortunately, Cournot did not define those economic problems which were not amenable to mathematical treatments. Surprisingly, this problem of profound significance has not received much attention during the past 160 years since Cournot. We will come back to this problem in section 1.7.

Perhaps the best exposition on the limitations of mathematical economics among classical economists is given by Alfred Marshall. Marshall who was the

[25]Jevons has referred to the following work: Nicholas F. Canard (1801) and William Whewell (1829, 1831, 1850, 1875).

Second Wrangler in the Mathematical Tripos in 1865 when he was studying mathematics at Cambridge, believed that there is a tendency in mathematical economics to emphasize those points which will fit easier into mathematical methods. He strongly warned economists on this unbalanced treatment of economic issues. In his *Principles* [(1890), pages 850-1] he wrote as follows: "And hence arises a tendency towards assigning wrong proportions to economic forces; those elements being most emphasized which led themselves most easily to analytical methods ... It is a danger which more than any other the economist must have in mind at every turn. But to avoid it altogether, would be to abandon the chief means of scientific progress".

For Marshall, explicit and clear economic meaning together with a potentiality of explaining economic observations were the two conditions for successful applications of mathematics to economics. Being very conservative in using mathematical symbols in economic analysis, he wrote in 1906 that "I never read mathematics now; in fact I have forgotten even how to integrate a good many things. But I know I had a growing feeling in the later years of my work at the subject that a good mathematical theorem dealing with economic hypothesis was very unlikely to be good economics; and I went more and more on the rule - (1) Use mathematics as a short-hand language, rather than as an engine of inquiry, (2) Keep to them till you have done, (3) Translate into English, (4) Then illustrate by examples that are important in real life, (5) Burn the mathematics, (6) If you can't succeed in (4), burn (3). This last I did often" (page 776, vol. 2, *Notes*). Let us now examine the limitations of modern mathematical economics in the light of Marshall's main condition.

Can Mathematical Methods Discover Economic Truths?

The most controversial issue in mathematization of economics is whether economic truths are discoverable through the instrumentality of mathematics? Recall that Marshall held the view that mathematics was only a short-hand language, and not an engine of inquiry. In this section we argue that in contrary to Marshall's view, mathematics can be an engine of inquiry in modern mathematical economics simply because there are economic questions which can only be answered by specific mathematical methods. This is where mathematical methods can really contribute to economic analysis.

Some Nobel prizes are awarded in economics mainly for demonstrating how the applications of some mathematical methods can provide useful economic results which otherwise could not have been obtained. For example, the first Nobel prize in economics was awarded to Ragnar Frisch and Jan Tinbergen in 1969 for "having developed and applied dynamic models for the analysis of economic processes". For the same reason, there are Nobel prizes in economics which are awarded to mathematicians. Leonid Kantorovich, a Russian mathematician, won the Nobel prize in economics in 1975 because he developed the mathematical theory of linear programming and for applying it to economic problems of optimum allocation of resources. Gerard Debreu, a French mathematician, is another example who won the Nobel prize in economics in 1983. In addition to contributions resulting from the direct applications of mathematics to specific economic problems, there are economic results whose discoveries are indirectly due to mathematics. These results also signify the real contributions of mathematical methods. Besides, the mathematical approach to economic problems can basically give economists a precise view of the subject which can be conducive to economic discoveries of significant value.

It is always easy and thus more common to start with some well-known mathematical method and then look for an economic problem which can be fitted into its mathematical structure. This approach is unlikely to provide any significant economic result. On the other hand, there might exist some well-known economic problems whose solutions necessarily require one or more of the following conditions: 1) familiarity of economists to some advanced mathematical techniques, 2) further advances in some of the existing mathematical methods, 3) discovery of some new mathematical theorems and methods. In this approach, mathematics can be the engine of inquiry and its applications might lead to *real contributions.*

Some economic examples for the three conditions listed above are as follows. Applications of optimal control theory to optimal allocation of economic resources over time is a good example for the first condition. von Neumann's *minimax theorem* (1944), which is the fundamental result in the theory of zero-sum two-person games, or the Kakutani's generalization of the *Brouwer's fixed-point theorem* (1941), both motivated by problems in economic game theory, provide examples for the second condition. And finally, Kantorovich's contribution to economics, mentioned earlier, is a clear example of the last condition. In fact, Kantorovich invented linear programming, the general mathematics of finite systems of linear inequalities, in order to solve problems in optimum allocation of resources. Since then, linear programming has been added to the literature of applied mathematics, in general and combinatoric analysis, in particular. Another example for the last condition is the simplex method in quadratic programming, invented by a mathematician, Philip Wolfe (1959), in order to solve problems in optimal investment decisions.

The invention of new mathematical methods or proving some new theorems for solving specific economic problems have led some writers to the conclusion that it is not only mathematics which has contributed to economic knowledge; contributions of economics to mathematics have also been significant. In this regard Debreu (1984) has wrongly concluded that examples such as von Neumann's minimax theorem or Kakutani's generalization of Brouwer's fixed-point theorem in pure mathematics discussed above, which were motivated by economic problems, are real contributions of economics to mathematics.

The misleading conclusion of Debreu (1984) stems from the fact that he does not recognize the existence of a conceptual asymmetry in the following two concepts: "contributions of mathematics to economics" and "contributions of economics to mathematics". The former is a real contribution: without which economic truths could not have been discovered; whereas in the latter, an economic problem acts only as a stimulus to mathematical discoveries and does not, in principle, constitute a real contribution since this stimulus could have been originated from other non-economic sources or even from a pure mathematical imagination.

Admitting mathematics as an engine of inquiry in economics might lead us to a pessimistic view on future directions of economic analysis. Mathematical innovations for solving economic problems fall, by definition, within the domain of the mathematical profession whose contributions might generate a new round of interesting mathematical problems which provide new grounds for further involvements of mathematicians in economics. This will certainly influence the direction of economic theorization by further widening the existing gap between theoretical economics and real economic life. As will be discussed below, the present situation in economic literature which is characterized by *too much* mathematics supports this view.

Grubel and Boland (1986) in an attitude survey of American economists and David Greenaway (1990) in the same survey for British economists report that the dominant view among economists is that there is too much mathematics in professional journals. More importantly, the majority believe that higher knowledge of mathematics does not necessarily imply better economics. Greenaway (1990) reports that 64.6 per cent of British economists have answered *no* to the following question: "Does the development of pure mathematical skills leave the economists adequately prepared for work in government or industry?" The similar figure for American economists' response as reported by Grubel and Boland (1986) is 61 per cent. It is interesting that the percentages of economists who have answered *yes* to the above question are 16.7 per cent and 9 per cent among British and American economists, respectively.

To complete the mechanism of growth generating cycle in modern mathematical economics, the institutional arrangements should also be taken into consideration. The rapid pace of industrialization and economic growth in the 1950's and 1960's and the increasing popularity of computer programming in modelling and planning in industry and trade, produced a greater optimism in mathematical economics and econometrics. The concomitant increasing involvements of mathematical economists and econometricians in governing bodies of academic research and training institutions as well as Governmental research departments have made it more convenient for mathematical economists and econometricians not only to publish easier than non-mathematical economists but to obtain jobs more conveniently. This new institutional setting has produced higher incentives among young economic scholars to study mathematically oriented economic studies, and will do so for the foreseeable future.

1.5 The Rocky Lane to Successful Co-ordination: The Development of the Relationship Between Mathematical Economics and Econometric

As discussed in the preceding section, the main objective of the Econometric Society, created in 1930, was to establish a successful partnership between economic theory, mathematical analysis and statistical methods. The co-ordinated research programmes recommended by the Cowles Commission and fully discussed in the meetings of the Econometric Society, were seen as effective instruments for the realisation of this objective. An examination of the research work during the past 65 years implies that establishing such a partnership has not been an easy task. The gap between mathematical theories and statistical representations of the real world economic problems is still considerably wide. This problem naturally falls in the domain of the interplay between theory and observation (or measurement) in quantitative economics which has attracted a great deal of attention during the past 15 years.[26]

The point which concerns me in this section and which has not yet received proper attention in the literature, is the nature of the disparity between mathematical economics and econometrics in building up models for empirical analysis. This will provide a background for the second Chapter (section 2.5) where a discussion of the critique of macro-econometric models is presented in the light of the ongoing debates on the relationship between theory and observation. In this connection, the rational expectations hypothesis, time-inconsistency and the Lucas critique together with the potential contributions of optimisation models are also discussed in Chapter 2 (sections 2.7 and 2.8).

[26]Leamer (1983, 1987), McAleer, Pagan and Volker (1985), Hendry (1987), Sims (1980, 1982, 1987), Morgan (1990), Wallis (1991), Pesaran and Smith (1995a, 1995b), among others, have addressed this issue extensively primarily from a methodological point of view.

The controversial philosophical issues in economic theorisation and in particular the interesting points arising from the interplay of theory and observation are extensively discussed in the methodology of economics, for example, O'Brien(1991). Let us start from the least controversial issue by saying broadly that an economic theory is essentially a consistent set of reasoning which coherently organises our *a priori* knowledge about an economic problem. The objective of the Cowles Commission's co-ordinated research programme in unifying the theory and measurement necessarily implied that such economic theories should be confirmed by data. This could best be done by designing a *mathematical model* which relates *theoretical* concepts to their associated *observable* counterparts. At this point, mathematical economics can establish a link with econometrics. Despite the 65 years of mutual co-operation towards this objective, the history reveals an unhappy partnership.

At the root of the subject lies the traditional dichotomy between the functions of mathematical economists and econometricians in building empirical models, in which the former are supposed to supply a mathematical model and the latter simply to estimate its parameters. The estimated parameters are then used for predicting endogenous variables which are of interest to policy-makers. The task of confronting theories with observations does not end with estimation or prediction; testing and evaluation of theories are also expected in this practice. In other words, mathematical economic theories, statistical techniques and data should necessarily be unified in an empirical model which can then be used for forecasting, policy and theoretical evaluations. It is the problems associated with such unification which have constituted the real impediments in the development of a closer partnership between mathematical economics and econometrics. Let us now examine the nature of these problems.

Formal mathematical economic models are conventionally supplied by mathematical economists who regard the coherence, rigour and generality (and not necessarily the explanatory power) of the model as the criteria for evaluating a good economic model. Econometricians, who are generally concerned with statistical techniques and data processing, value economic models by their ability to explain historical observations of the real economic facts. The major task of econometrics is, therefore, to transfer a mathematical economic model into an empirical model amenable to estimation and prediction. To the extent that econometric models are at variance with mathematical models the tensions between econometrics and mathematical economics accumulate. Some of these tensions are unavoidable and some can be resolved by gaining more insight into the nature of such divergence. It is fair for econometricians to maintain that purely theoretical models in their general forms are inappropriate for empirical analysis. Econometricians, therefore, endure the painful experience of model specification in which the inclusion of new explanatory variables, functional forms of behavioural equations, statistical properties of disturbance terms, lag structure and dynamic adjustment processes, expectation mechanisms, objective functions and the stochastic nature of behavioural equations as well as the measurement subsystem should all be fully examined.

Mathematical economists can express their discontent with the econometricians' choice of assumptions such as linearity in some essentially non-linear economic dynamics, white noise processes for disturbance terms, quadratic performance measures and with the limitations of econometric techniques in handling some problems of vital economic importance such as misspecification errors, the rational expectations, time-inconsistencies and estimating the shadow prices (or costate variables) in dynamic intertemporal optimisations.[27] Such a complex con-

[27]These shortcomings are discussed in Chapter 2.

struct makes it extremely difficult, from a statistical point of view, to test the validity of an econometric model. When an econometric model is rejected by historical data, it is difficult to identify whether the failure is attributable to the inherent inconsistency of the core theoretical model or to the auxiliary assumptions.

The errors associated with the economic data used by econometricians in parameter estimations add a new dimension of tension between mathematical economics and econometrics. Except for special cases where precise direct observations are possible (such as prices and quantities in financial markets), the majority of available data which are compiled mainly for administrative or business purposes (and not necessarily as a response to theoretical model requirements) are being seriously affected by measurement errors. Such errors, which are widely discussed in the standard econometric literature, can best be classified as errors of aggregation, imputation and sampling. It is well-known that any aggregation (over time, products or individuals) necessarily involves a loss of information. Imputation of non-market activities cannot be performed without considerable major errors. Finally, sampling errors cannot be dealt with successfully unless the statistics of the whole population become available. Under such circumstances, it is hard to accept the existence of a precise data set which corresponds to theoretical concepts such as capital, labour or intermediate goods. The adverse effects of data deficiency in econometric modelling are further exacerbated by the problems associated with measuring unobservable variables such as the general price level, the level of expectations or the expected inflation, which are usually of crucial importance in mathematical economic models.

1.5.1 The Turning Point in the Rocky Lane to Co-ordination: The Emergence of Alternative Strategies

As discussed above, the Cowles Commission approach to the unification of economic theory, statistical analysis and mathematical modelling could lead to the rejection of all theories, thus rendering the practice of econometric modelling a trial and error process. However, it was agreed until the 1970s that this approach, known today as the traditional strategy, was the proper way of doing empirical research work. The Cowles Commission's approach was discarded by a number of prominent econometricians and heavily criticised by most applied economists on the basis of the predictive failures of the traditional economic modelling together with the aforementioned shortcomings as well as the Lucas critique (Lucas 1976) that the policy invariant behavioural relations are inconsistent with dynamic optimisation.[28] The rocky lane to successful co-ordination thus entered its second phase of development in the 1970s with the arrival of a number of rival approaches.

The rival approaches all shared the idea that the remarkable academic investment in the formal econometric work, within the Cowles Commission's approach, had only marginal influence on the thinking about substantive questions in economics in general and macro-econometric modelling in particular. However, these approaches differ in many ways. At one extreme, Summers (1991, pp. 129-130) argues that "formal econometric work, where elaborate technique is used to apply theory to data or isolate the direction of causal relationships when they are not obvious *a priori*, virtually always fails ... Successful empirical research has been characterised by attempts to gauge the strength of associations rather than to estimate structural parameters, verbal characterisations of how causal relations might operate rather than explicit mathematical models and the skillful use of

[28]The Lucas critique, in the context of dynamic optimisation models, is discussed in Chapter two (sections 2.7 and 2.8).

carefully chosen natural experiments rather than sophisticated statistical technique to achieve identification". To prove his claims, Summers (1991, p. 130) invites the reader to "try and identify a single instance in which a 'deep structural parameter' has been estimated in a way that has affected the profession's beliefs about the nature of preferences of production technologies or to identify a meaningful hypothesis about economic behaviour that has fallen into disrepute because of a formal statistical test". Summers' strong criticism of the statistical and econometric approach to finding economic truths reminds us of Cairnes' strong position against the mathematical approach to economic analysis expressed 116 years earlier: "So far as I can see, economic truths are not discoverable through the instrumentality of mathematics. If this view be unsound, there is at hand an easy means of refutation- the production of an economic truth, not before known, which has been thus arrived at; but I am not aware that up to the present any such evidence has been furnished of the efficiency of the mathematical methods" (Cairnes, 1875, preface).

At the other extreme, there are approaches which strongly advocate the atheoretical data-based methods in applied economic research, most outstanding are the methods of Sims and Hendry. Sims (1980) argues that proper identification of the structural equations is impossible. The estimation of reduced forms by vector autoregression (VAR) is the most that an applied econometrician can hope for in practice. According to Hendry's approach (Hendry 1982), known as the *Error Correction Models* (ECMs), econometric models are created as approximations to an inherently unknown data-generating process which has to meet predetermined design criteria. Economic theory, data and measurement systems are used in designing the criteria. Although the closest of all the alternative methods to the Cowles Commission's approach to applied econometric analysis is the Hendry method, closing the gap between theory-based and data-based econometric approaches cannot be easily achieved. We will discuss this point further in Chapter 2 where Smith and Pesaran's emphasis on the role of economic theory in the unification of theory and empirical work as well as the debate initiated by the *Economic Journal* on this topic (November 1995) are critically examined.

The Cowles Commission's traditional strategy and the Econometric Society have opened a new avenue of research work in applied econometric analysis since the 1930s. This tradition is now under heavy attack by a number of rival approaches. The fact that the role of economic theory is not yet very clear in this debate has made the choice between competing strategies a difficult task for applied econometricians. As we will discuss in Chapter 2, despite all the ambiguities, the benefits resulting from this debate have shed more light on the Cowles Commission's aim of unifying theory, data and measurement. We can now see better. We can see further.

1.6 Economic Applications of Optimal Control Theory as an Illustration

In this section we concentrate only on *modern* optimal control theory. Discussions on economic applications of *classical* control theory are presented in the next chapter. The symbolic birth of modern control theory is the publication of Bellman's *Dynamic Programming* (1957) and Pontryagin's *Maximum Principle* (1955-59, its English translation in 1962) both developed largely by the requirements of space technology.

The emergence of optimal control theory as a new and powerful mathematical tool in pure and applied mathematics took place during the period when mathematical economics had been firmly established. There were not, therefore, in contrast to one or two generations earlier, any real and strong objections or resistance to applications of advanced mathematical methods in economic analysis. In fact, further mathematization of economics and utilisation of highly involved mathematical apparatus were warmly welcomed.[29]

In general, one can argue that the following could have been regarded as necessary conditions for successful application of optimal control theory to economic analysis.

i) The availability of interested mathematicians or control theorists to introduce the nature and significance of this new branch of mathematics to the community of mathematical economists.

ii) The knowledge about the appropriate areas for economic applications of optimal control theory.

iii) An understanding of the importance and limitations of such applications.

The fulfilment of the above three conditions could have produced a growth generating cycle in economic applications of optimal control theory.

Interestingly enough, the first condition was fulfilled by a mathematician who was one of the two inventors of optimal control theory, i.e. Richard Bellman. In the preface to his *Dynamic Programming* (1957) he clearly explained the method and its importance to economic analysis. After defining the subject matter of dynamic programming as mathematical theory of multi-stage decision processes, he wrote as follows: "The point we wish to make is that ... in economic, industrial, scientific and even political spheres, we are continually surrounded by multi-stage decision processes ... Unfortunately for the peace of mind of the economist, industrialist and engineer, the problems that have arisen in recent years in economic, industrial and engineering fields are too vast in portent and extent to be treated in the haphazard fashion that was permissible in a more leisurely bygone era ...

[29]Historically, it is interesting to note that the strong belief in contributions of mathematical methods to economics led Walras, 80 years before the emergence of control theory as a discipline, to predict the possibility of controlling an economic system. See, footnote number 13, above.

Whether [multi-stage decision processes] arise in the study of optimal inventory or stock control, or in an input-output analysis of a complex of interdependent industries ... or the study of logistics or investment policies ... they possess certain common thorny features which stretch the confines of conventional mathematical theory. It follows that new methods must be devised to meet the challenge of these new problems and to a mathematician nothing could be more pleased".

Bellman is perhaps the first mathematician who has viewed economics from a *state-space* point of view in system theory. In the second paragraph of the preface to his *Dynamic Programming* he stated the following: "Let us suppose that we have a physical system S whose state at any time t is specified by a vector p. If we are in an optimistic frame of mind we can visualise the components of p to be quite definite quantities such as Cartesian coordinates, or position and momentum coordinates ... or if we are considering an economic system, supply and demand, or stockpile and production capacities".

According to Bellman, optimal control theory, in the sense of dynamic programming, is one of the best examples in which pure mathematics has been developed in response to the real economic (as well as engineering or non-engineering) problems. Under such circumstances, one should expect *real* contributions from applications of optimal control theory to economics. Bellman, as a mathematician, had a clear idea of the potential applications of his method to economics. He has taken as his audience the following five groups: mathematicians, economists, statisticians, engineers and operations analysts. While he recommended chapters 1,2,3 and 9 of his book to engineer audiences, chapters 1, 2, 3, 5 and 9 were recommended to economists (*ibid*, preface, p. *xvii*). However, he has forgotten to recommend chapter 7 to economists also since this chapter, entitled *Bottleneck Problems*, is concerned with a multi-stage production process involving auto, steel and tool industries.

Despite Bellman's efforts to signify the importance and relevance of dynamic programming in solving economic decision problems, his method did not attract the attention of the community of economists at the time. This was mainly due to the nature of early developments in dynamic programming which was heavily dependent on computational algorithms and digital computers: "If we do not wish to suffer the usual atrophy of armchair philosophers, we must occasionally roll up our sleeves and do some spade-work. With the aid of dynamic programming and digital computers we can methodically engage in mathematical experimentations" [Bellman and Dreyfus (1962), p. *ix*]. This reminds us of Cournot (1838, p. 1) who wrote nearly 125 years before Bellman about the importance of mathematical experimentations in economic reasoning: " ... now the demand is for so-called 'positive' matter ... such as will throw the light of experience on the important questions which are being agitated before the country".

Bellman's conceptualisation of economic applications of dynamic programming fits much better to an econometric model designed for computing optimal economic policies rather than to mathematical models aimed at discovering the nature and characteristics of optimal policies. The inherent computational difficulties associated with the dynamic programming known as the *curse of dimensionality* together with the fact that econometric models of any practical value are usually medium to large scale, explain the fact that why the dynamic programming did not receive a warm welcome during the early stages of economic applications of optimal control theory. Despite Bellman's emphasis on the usefulness of the dynamic programming in economic analysis, the above factors persuaded control theorists and economists to examine closely the alternative but mathematically more sophisticated method of Pontryagin's maximum principle which is the modern version of the classical calculus of variations.

Applications of the classical calculus of variations to problems of economic depreciation [Hotelling (1925)], to economic studies in general [Evans (1928)], to optimal savings [Ramsey (1928)] and to the development of early models of economic growth [Solow (1956) and Swan (1956)] provided an appropriate background for a successful application of Pontryagin's maximum principle to optimal economic growth. Mathematical potentials of the maximum principle in identifying the properties of optimal paths of state and control variables made this method a high powered machinery in deriving the optimality conditions in models of economic growth.

To demonstrate the perfect compatibility of the maximum principle to theories of optimal growth, Dorfman (1969) stated that "optimal control theory is formally identical with capital theory, and that its main insights can be attained by strictly economic reasoning". To justify this, Dorfman started with a well-known problem in capital theory, i.e. a firm that wishes to maximise its total profit over some period of time with a given initial stock of capital. The rate of profit per unit of time depends on the initial condition and the decision taken by the firm. Maximization of the total profit earned from initial date to some terminal date will be a function of the entire time path of decision variables. Although the firm is almost free to choose the time path of policy variables it cannot arbitrarily select the amount of capital at each period since the latter is a function of the policies taken earlier. The firm is thus facing a policy formulation problem in a dynamic context whose solution is best provided by the maximum principle. Using this economic example, Dorfman obtained the necessary optimality conditions in the maximum principle, which provided an interesting economic interpretation of optimal control theory. This provides an excellent example of the real economic contribution of the maximum principle.

As we will show in Chapter 2, optimal growth theory, as a discipline in mathematical economics, could not have been developed without direct applications of the maximum principle. This clearly demonstrates the fact that optimal control as a *mathematical method* has significantly contributed to economic analysis. Moreover, we will show in Chapter 3 how the applications of the dynamic programming and the maximum principle can contribute to the literature on consumer behaviour under liquidity constraints.

1.6.1 Limitations of Economic Applications of Optimal Control Theory

The nature of economic problems whose solutions demand the application of optimal control theory identifies the limitations of such applications. As discussed earlier, whenever the solution of an economic problem motivates a researcher to find an appropriate mathematical method, results of value might be expected, but not vice-versa. Engineering optimal control theory has been developed in the 1950's largely by the requirements of space technology. The potentiality of this method for physical systems and particularly for automation is remarkable. The fact that the advances in optimal control theory have been motivated by automatic control of physical systems identifies the underlying limitations in its economic applications.

If we broadly classify the literature in this field into two categories of mathematical and engineering optimal control theory, the potential economic contributions are expected to come from the former. Even centrally planned economies, who might theoretically admit strict central economic controls, cannot rely on automatic economic controls since the underlying economic models are in sharp contrast to physical systems. An examination of the literature in economic applications of optimal control indicates that the engineering approach to economic applications of optimal control aiming at automatic design and computations of optimal trajectories have been progressively replaced by mathematical approach which aims at identifying economic optimality conditions. Let us examine this point further.

The decades of the 1960's and 1970's were the active periods in applications of optimal control to models of optimal growth as well as to econometric models. The author has reported elsewhere [Derakhshan (1978)] that during these two decades the number of papers on economic applications of optimal control, published mainly in engineering journals, were 347 as compared with 400 such papers which appeared in different journals in economics. The aim of a considerable number of papers published in engineering journals was to use the idea of automatic control in modelling industries (or the macroeconomy) in order to compute the optimal state and control trajectories. This situation has considerably changed during the past decade.

Many leading engineering journals such as *Automatica, IEEE Transactions on Automatic Control, International Journal of Control, International Journal of Systems Sciences, SIAM Journal of Control*, among the 38 engineering and mathematical journals reviewed in Derakhshan (1978), once very active in publishing papers on economic applications of control theory, have either completely abandoned or rarely publish such work. Moreover, a careful inspection of papers published in the *Journal of Economic Dynamics and Control*,[30] the new

[30]In 1978, the National Bureau of Economic Research (NBER) decided to discontinue publications of the *Annals of Economic and Social Measurement* which usually published selected papers from conferences and annual meetings on economic applications of optimal control theory. Shortly thereafter, North-Holland Publishing Company agreed to the establishment of the *Journal of Economic Dynamics and Control* to continue the publication of papers on economic applications of optimal control in a wider context. At the same time, the *Society for Economic Dynamics and Control* was organised to promote and sponsor international conferences and research projects in this field. The first issue of this journal appeared in February 1979.

specialised journal on theoretical and applied work on economic applications of control theory, reveals the fact that almost all the research work published in this journal during the last ten years have been more theoretical in nature with no serious attempts in computations of optimal economic trajectories within an automatic control framework.

1.7 The Logic of Abstraction: The Origin of Limitations in Mathematization of Economics and its Implications for Optimal Control Applications

We discussed in sections 1.4.2 and 1.6.1 the limitations of mathematical economics in general and economic applications of optimal control theory as an example. In this section, we examine the origin and nature of such limitations. Recall that Marshall (1890) warned economists of the danger of an unbalanced treatment of economic issues by a mathematical approach on the ground that those economic elements being most emphasized by mathematical economists which usually led themselves more easily to mathematical methods (see section 1.4.2). Marshall's recommendation is rather vague and cannot constitute a programme of achieving higher degrees of precision in theorization of mathematical economics. A question of prime theoretical importance is what economists *should do* to avoid this danger? We can put this question alternatively as follows: If economic applications of *engineering* control theory have not been promising due to the underlying differences between physical and economic systems, why have mathematical theories of control for *social systems* in general and for *economic systems* in particular not yet been developed? To what extent this has been due to the fact that mathematicians were not well acquainted with social or economic sciences or social scientists were not good mathematicians? Our analysis in previous sections implies that none of these can provide a satisfactory answer. It seems that the logic of abstraction in methodology of economic theorization must play the key role.

To avoid unnecessary involvements in methodological issues which are beyond the scope of present work, we confine the argument to very simple heuristic assertions. Without going into details of *defining* mathematical reasoning, it is easy to see that mathematical machinery is a system of logical reasoning based on *abstract* notions. No single topic in economics can be treated mathematically without first being reduced to abstract and narrow concepts and then being fed into the mathematical machinery in order to infer necessary logical conclusions. Since economic input to mathematical machinery is abstract, the output will also become abstract.

Under what conditions can one obtain economic results of any significant value using mathematical reasoning? To answer this question, I first assume that the value of a result depends on its explanatory power: either being useful in explaining some other unknown theoretical economic facts, or being able to explain a real world economic observation. The former is a contribution to pure or mathematical economics and the latter constitutes contributions to applied economics.

An examination of the classical work on mathematical economics (Cournot, Jevons, Walras and Pareto), reveals the fact that their ultimate goals were to discover the dynamics of "pure economics". However, they had some reservations on the unconstrained advances in pure or mathematical economics. If one favours unbounded developments in mathematical economics, as many of the contemporary mathematical economists do, there would be endlessly fascinating theoretical journeys in mathematical economics with no necessarily direct references to real world economic issues. An inspection of articles published in the specialised *Journal of Mathematical Economics* might support this hypothesis.

The main task is, therefore, to identify the conditions under which mathematization of economics can lead to results of significant value in explaining real world economic issues. We claim that the methodology employed in abstracting notions from real world economic life can, in principle, provide an answer. We know that a real economic system cannot be studied in isolation of the related historical, political, sociological and cultural systems. The attempts by Cournot, Jevons and Walras in creating mathematical economics to study economics in isolation of other related systems, were the first major move in a wrong direction in the process of abstraction. Further, abstractions within mathematical economics, together with more simplified assumptions which were made to facilitate the applications of more advanced mathematical methods, have produced the existing rich and yet abstract, literature in mathematical economics.

The origin of this unfortunate outcome lies in the erroneous method of abstraction employed by classical mathematical economists. Their method of abstraction was nothing more than a simple division of multi-dimensional political economy into different disciplines: "Political economy is in a chaotic state at present, because there is need for subdividing a too extensive sphere of knowledge" [Jevons (1871), p. 20]. It seems unlikely that the results obtained from the behaviour of a fragmented part of a multi-dimensional political economy can truly represent the behaviour of the system as a whole. The results obtained in one-dimensional mathematical economics are valid only within its own domain and cannot by itself provide results of value for a real multi-dimensional world of economic life. The following example might clarify this point.

Consider a sphere in a three dimensional space. One can study this object by examining the properties of its different fragmented parts. Evidently, the results obtained in this way cannot provide an understanding of the properties of the sphere. Alternatively, one can reduce, or project, the sphere on each of the three two-dimensional planes. The resulting circles are the *reduced versions* of the sphere; and from the properties of circles one can infer the properties of sphere as long as the *interrelationships* among projected circles from one hand and the *relationship* between projected circles and the sphere on the other hand are known. A proper method of abstraction should therefore reduce complexities while preserving the underlying properties of relations.

What would be the implications of the above argument for mathematical economics in general and optimal control applications in particular? In the absence of a well-defined method of abstraction, one can claim that applications of mathematical methods to economic analysis are most promising in those areas where the abstract economic notions, to be used in mathematical machinery, constitute close approximations to economic realities. In these cases, most likely to be found in technical questions in microeconomics, the effects of the underlying non-economical factors, such as political, cultural, and historical elements, are minimal. Formulations of efficient resource allocation for which Kantorovich (1939) developed the method of linear programming, or the inter-industry models for which Leontief (1941) invented the input-output technique, fall within this category. This explains why these examples are always referred to as successful applications of mathematics to economics.

Using the above criterion, one can argue that the applications of optimal control theory to economic policy formulations, particularly at macro-level, cannot be very successful. This is due to the nature of policy oriented econometric models as well as the properties of optimal control methods. A useful macro-econometric model is usually a non-linear medium to large scale model, whereas optimal control methods, as will be discussed in Chapter 2, are well developed for linear models of a small size. Moreover, analytical solutions in linear systems exist for optimal control paths when the objective function is quadratic, whereas this class of objective functions does not satisfactorily explain economic realities since it fails to differentiate positive and negative deviations from economic targets. Despite the current validity of these objections, one can truly argue that these are technical problems in nature and advances in econometric modelling and control techniques are expected to relax them in due course.

There remains another more serious technical barrier to the applications of optimal control theory to economic policy formulations. Recall that due to the specification errors and errors of approximations inherent in econometric models, simulations with these models are usually adjusted by experts' guess and round table discussions before practical policy formulations are recommended. We will discuss in Chapter 2 that the nature of automatic simulations and forecasting in feedback loops in optimal control algorithms do not usually provide an opportunity for experts' guesses or qualitative analysis which lead to quantitative adjustments. This explains why the high optimism in building up optimal macro-econometric control models in the early 1970's has now significantly decreased. In fact, the few research work done since the 1970's on optimal control applications in macro-econometric models have been carried out mainly for an examination of properties of models and the relations between alternative policy objectives rather than for deriving policy recommendations.[31]

Problems associated with the rational expectations in consumers' as well as producers' behaviour will add a new dimension of difficulties in economic appli-

[31]For the case of UK, see Melliss (1984) who has reported some experiments with optimal control on the Treasury Model. Work is also done by PROPE team (Programme of Research into Optimal Policy Evaluation) at Imperial College. For example, see Currie and Karakitsos (1981) and Brooks, Henry and Karakitsos (1983) on NIESR model. See, also Artis and Karakitsos (1982) and Henry, Karakitsos and Savage (1982). Van Der Pleog (1982) reports his control experiments with Cambridge growth project model.

cations of optimal control theory.[32] Moreover, if we take into account that the underlying econometric model cannot capture, due to the nature of abstractions in mathematical economics, the effects of existing qualitative feedbacks between Government's structure , producers, consumers, financial institutions, legal systems, etc., within a socio-cultural and political environment, then the approximation error in the underlying mathematical model becomes so severe which makes the numerical optimization totally unreliable.

On the basis of structural errors of approximations in mathematical modelling of the economy and the nature of optimal control methods discussed above, the real contributions of optimal control theory in economic analysis are expected to be in mathematical economics where it can freely use abstract notions in order to discover the behavioural optimality conditions. By applying optimal control theory to deterministic and stochastic consumption behaviour of an individual under liquidity constraints, I have shown in Chapter 3 that significant new results on optimal behaviour of consumers can be derived. And, finally, applications of optimal control theory to dynamic Leontief models in Chapter 4 have produced new interesting results which further support the hypothesis developed in this chapter.

1.8 Summary and Concluding Remarks

The analysis of the origin and limitations of mathematical economics can illuminate the nature and boundaries of economic applications of optimal control. An historical approach has been adopted in this chapter while attempts are made to avoid methodological issues.

The nature and limitations of mathematical economics are examined in terms

[32]See Chapter 2, Sections 2.8 and 2.9.

of what I have classified as classical and modern mathematical economics. The origin and sources of classical mathematical economics are studied in the light of the following questions: 1) Why none of the work on mathematical economics published before Cournot was given the distinction of being the beginning of mathematical economics? 2) Why Cournot's work was totally forgotten for over 30 years until Jevons (1871) revived it? 3) Why Jevons greatly esteemed Cournot's contribution while not being able to understand him thoroughly? 4) Why masters of classical economics did not become interested in mathematical economics? To answer these questions, four hypotheses are critically examined before I present my own hypothesis.

Debreu's incidentality hypothesis of early developments in mathematical economics, which has been shared by some economic historians like Robertson (1949), Theocharis (1983) and Gherity (1990), is found to be inadequate since it has failed to be consistent with the views clearly expressed by pioneers of classical mathematical economics such as Cournot, Jevons and Walras. Moreover, Debreu's argumentation in supporting his hypothesis has happened to include a number of erroneous historical records.

As regards why the early developments in mathematical economics were totally ignored by classical economists, Cournot's hypothesis (1838), which is supported by Fisher (1891), is examined. According to this hypothesis there are two possible explanations: *a*) erroneous presentations of mathematical economics and *b*) the poor mathematical knowledge among classical economists. We have found evidence contrary to this hypothesis.

Walras's hypothesis, i.e. the "narrowness of ideas", which gave much emphasis on inductive reasoning in economics cannot be accepted on the ground that the revival of mathematical economics by Jevons, Walras and Pareto was not accompanied by any serious interest in measurements and experimentations in economic analysis. We have shown that Walras's own work can be regarded as contributions to deductive reasoning with the effect of widening the gap between pure theorization and economic experimentation.

According to the hypothesis of von Neumann and Morgenstern (1944), the "unfavourable circumstances" resulting from the following factors produced the general lack of interest on mathematical economics among classical economists: the vagueness of basic economic concepts; the inadequate empirical economic facts; and limitations in mathematical treatment of human behaviour. Despite the fact that these factors are individually true, the lack of interest in mathematical economics has had nothing to do with the combination of these factors. On the contrary, it is shown that von Neumann and Morgenstern's argument will direct us to the conclusion that real developments in mathematical economics necessarily require discoveries of new mathematical methods for social sciences; and until then the science of economics can provide no recommendation for factual economic management.

Our hypothesis of one-dimensionalisation of economic analysis is discussed in section 1.2.5. According to this hypothesis, classical mathematical economics emerged in response to a growing desire for one-dimensionalisation of economic analysis which was equivalent to reduction of classical political economy to a mechanical economic science. Advances in classical and Marxian economics had produced a state of multi-dimensionality in economic analysis in which economic issues were studied in connection with historical, political, social, and legal systems. "Mechanical-scientific" approach resulting from the rapid advances in physical sciences during the 19th century and the "non-political" approach resulting from reactions to rapid advances in Marxism are considered in this chapter to be the main interrelated responses to such environment of multidimensionality in economic analysis.

In the analysis of mathematical economics and the formation of mechanical economic science, in section 1.2.5, we have examined the following two topics: the nature of classical mathematical economics, and mathematical economics as a remedy to multi-dimensional political economy. The argument in this section is carried out with direct reference to the work of Cournot, Jevons and Walras. It is shown that a successful diffusion of Jevons's and Walras's mathematical economics necessarily required a revival of Cournot (1838) together with the 38 work published on mathematical economics before Cournot, i.e. during the period 1711-1838 and the 62 such work published from Cournot to Jevons. This could have established an army of supportive literature for the new discipline of mathematical economics. It is not, therefore, surprising that Jevons praised Cournot without fully understanding him.

Mathematical economics as a response to Marxian economics is another hypothesis which I have examined in section 1.2.5. This hypothesis has emerged from the chronological coincidence of proliferation of Marxian literature and the emergence of mathematical economics from one hand and the impassive and negative attitudes of pioneers of mathematical economics towards contributions in Marxian economics. We have shown that the subdivisionism, so strongly advocated by Jevons and Walras, as a remedy to the so-called "chaotic state of multi-dimensionality" in economics, was successful in making Marxian machinery futile or irrelevant within the context of pure or mathematical economics. By concentrating only on the economic dimension, and ignoring the feedback mechanism and structural dependencies with political, historical, legal and social aspects, mathematical economics necessarily depleted the real content of fundamental economic concepts and thus downgraded them to abstract notions suitable for mathematical manipulation.

On the basis of arguments presented in section 1.2.5, we have not been able to accept the hypothesis that mathematical economics has emerged as a response to Marxism. However, the fact of the matter is that the structure and methodology of mathematical economics made it possible to leave the whole body of Marxian economics to one side. This is exactly what Jevons and Walras and most of their disciples, have effectively done: despite having long discussions on sensitive issues like "value", they were successful in managing their arguments without even a single reference to Marx or his work on this topic.

Having examined the nature and origin of classical mathematical economics, we have then analysed the origin and formation of modern mathematical economics in section 1.3. We could not identify a course of continual developments in mathematical economics from the time of Pareto (1897) and Marshall (1890) to the 1930's. Different individual contributions can be considered either as attempts to reorganise and consolidate previous mathematical treatments of economics or to provide new and comprehensive reports on the significance of the already known results. Bowley, Evans and Wicksell are studied in this connection. It seems that during this period the community of economists had remained faithful to their non-mathematical Marshallian type economic analysis.

According to Debreu (1986), among others, the publication of *The Theory of Games and Economic Behaviour* by von Neumann and Morgenstern in 1944 marks the beginning of modern mathematical economics since this book induced a new level of logical rigour in economic reasoning and formulated, for the first time, a new mathematical method for economic analysis, i.e. the mathematical theory of games of strategy. We have refuted this hypothesis in section 1.3.1.

We have proposed the hypothesis of *coordinated research programmes* as the origin of modern mathematical economics. In section 1.3.2 we have first established the absence of any coordinated research work in mathematical economics prior to the 1930's. I have claimed that the creation of the *Econometric Society* on December 29, 1930 and the *Cowles Commission for Research in Economics* in 1932, with the prime objectives of unifying economic theory, mathematical analysis, and statistics and promoting and encouraging research work in this direction, have marked the beginning of modern mathematical economics. It was clearly recognised by the Econometric Society that the mutual penetration of quantitative economic theory and statistical observations, can profoundly change the tradition of economic theorization in classical mathematical economics.

To comprehend the salient features of modern mathematical economics, a comparative analysis of classical and modern mathematical economics is presented in section 1.4. In this regard, the attitudes of mathematical economists towards non-mathematical economists and their perceptions on the limitations of mathematization of economics are examined.

An examination of classical contributions in mathematical economics reveals the fact that classical mathematical economists had, more or less, an inimical attitude towards non-mathematical classical economists. On the other hand, masters of classical political economy usually expressed their scepticism on mathematical economics on the ground that either economic truths cannot be discovered through the instrumentality of mathematics, or the subject matter of mathematical economics does not possess an independent value of its own for solving factual economic issues. However, this feature is shown to be basically absent among contemporary mathematical economists. In fact, mathematical economics no longer fights for the right to live, it dictates the survival conditions of other disciplines in economic theorization.

We have discussed in section 1.4.2 that classical mathematical economists generally accepted the view that to be explicitly mathematical does not ensure the attainment of economic truths; only those mathematical applications are admissible which produce results of value. Despite the ambiguity of the term "results of value", we have shown a wide range of diversities among classical mathematical and non-mathematical economists on the concepts of "limitations" and "results of value" in economic applications of mathematical methods.

In contrast to Marshall's view that mathematics is not an engine of inquiry but is only a short-hand language, we have established in section 1.4.2 that mathematics has proved to be an engine of inquiry in modern mathematical economics since there are *economic questions* which can only be attended by specific mathematical methods. We have tried in this section to identify the conditions under which mathematical methods can really contribute to economic analysis. Some writers, like Debreu, has exaggerated the contributions of mathematical methods to economics to the extent that contributions of economics to mathematics have also been regarded as significant. We have shown that Debreu's hypothesis can be refuted.

An examination of the research work during the past 65 years reveals the fact that the efforts of the Econometric Society and the Cowles Commission to establish a successful partnership between economic theory, mathematical analysis and statistical methods have not been very successful. The gap between mathematical theories and statistical representations of the real world economic problems is still wide. Section 1.5 examines the underlying factors in the failure of designing a mathematical model which relates *theoretical* concepts to their associated *observable* counterparts. It is shown that when an econometric model is rejected by historical data, it is difficult to identify whether the failure is attributable to the inherent inconsistency of the core theoretical model or to the auxiliary assumptions.

Section 1.5.1 examines the emergence of alternative strategies, i.e. theory-based and data-based econometric approaches, to the Cowles Commission's traditional approach to economic modelling. The relationship between theory and observation is further discussed in Chapter 2 (section 2.5) where Smith and Pesaran's emphasis on the role of economic theory in unification of theory and empirical work as well as the debate initiated by the *Economic Journal* on this issue are critically studied. Section 1.5.1 concludes that the Cowles Commission's traditional strategy has opened a new avenue of research work in applied econometric analysis since the 1930's. The ambiguities of the role of economic theory in the relationship between theory and observation has made the choice between the competing strategies a difficult task for applied econometricians. However, the Cowles Commission's approach has significantly benefited from the ongoing debate on alternative strategies.

The analysis of the origin and limitations of mathematical economics presented in sections 1.2 to 1.4 and the problems associated with the development of the relationship between mathematical economics and econometrics discussed in section 1.5 constitute the analytical framework for our discussion in section 1.6 on economic applications of optimal control theory as a mathematical method. We have shown in this section that all the possible conducive factors in such applications were historically available. By the late 1950's and early 1960's modern mathematical economics was firmly established in the community of economists and further mathematization of economics and utilisation of more advanced mathematical techniques were generally recommended. Bellman himself explained the relevance and significance of his dynamic programming applications to economic analysis when he announced his powerful optimal control method to the community of mathematicians and control theorists.

Advances in theoretical economic growth in the 1950's and the compatibility of growth models with optimal control applications made models of economic growth an appropriate field of economic control applications. It is shown in section 1.5 that the weakness and shortcomings of the dynamic programming applications to economic growth models in the 1960's and 1970's do not follow from theoretical or conceptual limitations; the inherent computational difficulties associated with the dynamic programming explains such partial failures. We have shown that Pontryagin's maximum principle was proved to be more useful in identifying the properties of optimal trajectories in growth models. As we will discuss in Chapter 2, contributions of Pontryagin's maximum principle in the formation of optimal economic growth theory is profoundly significant. This demonstrates the real contribution of optimal control as a *mathematical method* in advancing economic analysis.

Limitations of economic applications of optimal control theory as a mathematical method is presented in section 1.6.1. We have argued that whenever the solution of an economic problem has motivated a researcher to find an appropriate mathematical method, useful results would have been expected. The fact that advances in optimal control theory have been motivated by automatic control of physical systems identifies the underlying limitations in its economic applications. We have demonstrated that the *engineering approach* to economic applications of optimal control aiming at automatic design and computations of optimal trajectories have been progressively replaced by the *mathematical approach* which aims at identifying economic optimality conditions.

It is argued in section 1.7 that the origin of partial failure in economic applications of optimal control theory lies in the erroneous method of abstraction in mathematical economics. Economic systems are more complex than physical systems since they are inter-related to social, historical, legal and political systems. Since mathematical machinery is a system of logical reasoning based on abstract notions, any applications of optimal control theory requires the abstract concepts derived both from economic models to be controlled and from the objective functions to be maximised. A proper method of abstraction should, therefore, reduce complexities while preserving the underlying properties of relations between different subsystems. In the absence of a mathematical method ensuring the above criterion, we have shown in section 1.7 that the applications of optimal control theory to economic policy formulations are most promising in those areas where the abstract economic notions, to be used in mathematical machinery, constitute close approximations to economic realities.

Chapter Two

Control Theory and Economic Policy Optimisation: Developments, Challenges and Prospects

2.1 Introduction

On 18 July 1951, an informal evening session took place during the Conference on Automatic Control at the College of Aeronautics, Cranfield. The purpose of this session was to bring to the attention of the conference the analogy between problems arising in stabilising an economic system and those of physical systems; with the implication that communities of economists and engineers could benefit from their respective specialisations [see Tustin(1952)]. Richard Stone from Cambridge opened the session with a lecture in which he referred to the use of electrical analogues in interpreting Leontief transaction matrices and demonstrated the similarity of Kirchhoff's first law to the accountancy relationships for a basic unit. Tustin, Professor of electrical engineering at the University of Birmingham, showed how dynamic economic models being used by econometricians corresponded precisely with the engineer's *scheme of dependence*. He interpreted the Keynesian economic system in terms of a closed sequence with the multiplier relationship as the effect of a feedback. The session concluded that economists might profit from the work of control engineers in making an economic system work as a regulator to maintain full employment without inflation; and that the cooperation of control engineers and economists would be both practical and useful.

Twenty-seven years later (1951-1978) and after the publication of about 1400 research work on applications of systems and control theory to economic analysis [see Derakhshan (1978)], the *Committee on Policy Optimization* chaired by Professor Ball, of the London Business School, published their report in March 1978, the purpose of which was

> *...to consider the present state of the development of optimal control techniques as applied to macro-economic policy. To make recommendations concerning the feasibility and value of applying these techniques within Her Majesty's Treasury.*[33]

The Committee concluded that

> *the application of optimal control to the analysis of economic policy is feasible and, applied at working level to the generation of simulations and as a means of testing the properties of economic models, it is likely to be of value. We are not, however, able to say that this is the single most important priority in the development of modelling and forecasting practice.*[34]

Developments of control theory applications to economic analysis thereafter, i.e. during the period of 1978-1996, has not refuted the prediction made by Ball's report but at the same time has opened new promising avenues of research work in this field resulting from advances in theoretical economics such as the rational expectations hypothesis. Our objectives in this chapter are: *i*) to explain the evolution of the mainstream applications of control theory to economic policy optimization and to examine its sources and the underlying factors at work in

[33]See Ball (1978), p. 1.
[34]See Ball (1978), p. 113.

such developments; *ii*) to identify the most successful areas of control theory applications in the field of economic policy optimization; and *iii*) to provide a basis for analyzing the limitations in such applications and the prospects for further research. We will constrain our study to applications of control theory to economic modelling and policy optimization processes.

Most definitions of economics share the idea that economic analysis deals with the allocation of given means for the optimum satisfaction of given ends. In this sense, an economic system can be regarded as a *closed system* with given means defined as a *bounded control space* and satisfaction represented by a *performance criterion.* From a mathematical point of view, the method of optimal control can, in principle, effectively solve this problem. More specifically, optimal growth theories as well as stabilisation policies possess the characteristics which make the application of optimal control theory more demanding. Optimal growth theory is concerned with the optimal choice among alternative trajectories along which an economic system can be transformed from a given initial position to a desired state at the end of a specified (or unspecified) horizon, where each trajectory is generated by applying a set of feasible controls. The theory of stabilisation policy deals with government actions in dampening unwanted fluctuations and at the same time driving an economic system along a desired path. According to modern optimal control theory, an admissible stabilising control should possess an optimising character. This has made the application of modern optimal control theory to economic growth and planning even more productive, for an economic stabilisation programme with no optimality condition may not guarantee an optimum design for an economic system.

The prime objective of this chapter is not to review the literature on economic applications of control theory. As mentioned above, our aim is to examine the incentives, achievements, failures and perspectives in cooperation between the two disciplines of control theory and optimal economic policy formulation. We hope that the material presented in this chapter can basically answer the following questions: How did control engineering emerge as a mathematical tool in the domain of economic policy design? What role has been played by theoretical advances in economic optimization in such developments? Has control theory been successful in contributing towards the theoretical basis in optimal economic policies? What have been the most significant contributions of control theory applications in the theory and practice of economic policy optimization? Has control theory proved to be of any profound significance beyond being an engineering tool in the economist's work-box? What have been the factors responsible for the failed integration of control engineering in the theory of economic policies despite 45 years of continuous research work and cooperation between control theorists and economists? Where have the applications of control theory failed and why? Are these failures amenable to technical solutions or they are deeply rooted in more fundamental theoretical differences existing between physical and economic systems? And finally, what are the most promising fields of future research work in this area?

Optimal control theory is basically a mathematical construction whose capacity to solve problems in economic policy optimization depends on its mathematical properties. Our methodology in analyzing the above mentioned questions is to examine the developmental pattern of economic control applications with reference to the mathematical structure of control theory. We try to identify those properties which have led (or will lead) to conceptual and practical contributions in economic policy optimisation.

Section 2.2 explains how, in the 1930's and 1940's, classical control theory was introduced to the communities of economists interested in policy recommendations for economic stabilisation. In section 2.3, we discuss how the 1950's advances in control theory, known as *modern optimal control*, solved many problems in optimal economic growth theory -an active and rapidly expanding branch of economics in the 1960's. The contributions of control theory to econometrics in the form of estimation and computation of optimal economic state and control trajectories -a popular field of research in the 1970's- are also discussed in section 2.3. In the late 1970's the entire land of basic optimal control theory were known to the community of mathematical economists and econometricians. In this regard, an examination of stochastic and adaptive control theory and their applications to economic policy optimisation is presented in section 2.4.

The question we posed in Chapter one (section 1.3.3) that what has been gained by the Cowles Commission's programme on unifying mathematical economic theory and measurement system has received wide attention during the past 10 years especially with regard to the macro-econometric modelling and advances in dynamic optimisation of the behaviour of a representative economic agent. Section 2.5 examines the core of the argument in this field. The analysis of the origin of the gap between data-instigated and theory-based econometric models are also discussed and the role of dynamic optimising models together with a critical examination of the attempts to bridge this gap are examined. A critical analysis of Smith and Pesaran's contribution, i.e. the importance of *economic theory* in tightening the link between theory and empirical work in optimising models is also included in this section.

The rational expectations hypothesis which significantly changed the way in which economic policies were perceived, provided a strong criticism of the application of control theory to economic policy optimisation. Economic agents respond usually not to the signals which are mechanically generated by the controller in an engineering type environment but to their own expectations of economic state variables. Rational forward-looking expectations, in contrast to the case where expectations are functions of the past behaviour, make serious difficulties in standard formulation of policy optimisation. Policies which are believed to be optimal *ad hoc* will become sub-optimal upon realization. This problem suggests the utilisation of dynamic game theory between the controller and the agent. Section 2.6 considers the role of the rational expectations, the Lucas critique and the policy ineffectiveness debate in economic applications of optimal control theory.

Since the standard dynamic programming does not accommodate the impact of future policies on current values of state variable, the principle of optimality breaks down for optimal control of non-causal or forward-looking models in which current state variables depend on the anticipated future states. Section 2.7 examines the very important problem of time-inconsistency in the optimal control of macro-econometric models with rational expectations. This section also considers the impact of *reputation* and the stochastic environment on the problem of inconsistency in dynamic choice. The interesting question of how can the recent developments in optimal control of macroeconomic models with forward-looking expectations contribute to the practice of econometric model building is another topic which is discussed in section 2.7. And finally, section 2.8 provides a summary and a brief concluding remarks.

2.2 The Beginning: Classical Control Theory and Economic Stabilisation

From a mathematical point of view, the optimum regulation of an economic system towards attaining a desired objective falls within the domain of dynamic optimization, for the solution implies choosing the best combination of admissible instruments and applying them with the optimum solutions of timing and dosage. In this regard, control theory can be considered as the most advanced available method in optimization of a system's behaviour. Our starting point is to examine economic optimization problems prior to the classical and modern control theory applications.

We might refer to Ramsey (1928) as the first significant work which has applied the classical Euler-Lagrange method to solve an economic optimization problem. The question for Ramsey was how much of its income should a nation save? To answer this question a utility function was assumed and the application of Euler-Lagrange method produced the following simple rule: the rate of saving multiplied by the marginal utility of money should always be equal to the amount by which the total net rate of enjoyment of utility falls short of the maximum possible rate of enjoyment. Further advances in economic optimisation followed by the realization that a given required commodity bill can be produced by more than one process. It is exactly this element of choice between different processes, by which a required output level can be achieved, that has dominated the pattern of development in economic optimization problems.

The initial attempts to formulate the problem of choice in the dynamics of economic production were *A Model of General Equilibrium* by von Neumann (1938) and *Mathematical Methods in the Organizing and Planning of Production* by Kantorovich (1939). von Neumann considered a typical economic system in which goods are produced not only from *natural factors of production*, but in the first place, from each other. There may also be more technically possible processes of production than goods. The problem for von Neumann was to identify which processes will actually be used and which will not, i.e. being *unprofitable.* Solving such a problem will lead to a system of inequalities whose unique solution is not evident. The mathematical proof is possible only by means of a generalization of Brouwer's Fixed-Point Theorem, i.e. by the use of fundamental topological facts.

Kantorovich formulated a different problem but in the same context, i.e. achieving the highest possible production on the basis of the optimum utilisation of the existing reserves of industry, materials, labour and equipments. Increasing the efficiency of an enterprise or of the whole branch of an industry, can well be achieved by optimising production planning. The most important factor is considered to be the optimum distribution of the machine work and the best utilisation of raw materials, transportations and so on, which will lead to the formulation of the economic problem as an extremal problem in the field of mathematical programming.

In the 1930's, when the underlying theory of servo-mechanism in engineering was being established and when the discussions in terms of self-regulating systems and automatic stabilisation were in fashion, some economists began to study economic problems, such as cyclical behaviour and oscillations, in the context of self-regulating systems. The work of Frisch (1933) and Kalecki (1935) fall in this category. Most writers in this era, for example Keynes (1936), concluded that there is no tendency inherent in a capitalist economic system to generate stability and full employment; thus a control action in the form of government economic policies was necessary.

The work of Goodwin (1951a,1951b), Cooper (1951), Simon (1952), Tustin (1952) and Phillips (1954) mark the direct applications of *classical control theory* to economic analysis. Goodwin demonstrated that a servo-mechanism system regulates its behaviour by its own behaviour in the light of its objective. This explains why, for example, a human being usually succeeds in a complicated operation of picking up an object by minimising the distance between hand and object (a *tracking* problem). The work of Goodwin (1951) is of special importance as it is the earliest attempt in which an *error activated feedback* is applied to the analysis of market behaviour and business cycles. The applicability of a servo-mechanism to the theory of firm has been discussed by Cooper (1951). Simon (1952) studied very carefully the problem of controlling the rate of production on a single product in terms of servo-mechanism theory. He used the Laplace transformation method to examine the stability and the steady-state behaviour of his production control system. Tustin, who was an engineer, analysed the Keynesian model by control system theory and used the Nyquist criterion, Fourier analysis and Laplace transformation from control theory to analyse the possibility of stabilising the economy.

The work of Phillips (1954) was also concerned with the stabilisation of a closed economy. The government was seen as the main stabiliser, and three types of stabilisation policies were used. These policies, taken from control theory, were the proportional, integral and proportional plus integral techniques. He was specifically concerned with the question of to what extent government expenditures can be used as a controller to drive the economy along a desired trajectory, and in particular, to offset a deficiency in private demand while avoiding undesired fluctuations in output. Using the principles of servo-mechanism and feedback control theory, he demonstrated that in the usual multiplier-accelerator model, the time-path stability of the stabiliser (government expenditures in his example) differs for different types of economic policies. In Phillips' analysis, the full employment level of aggregate output is taken as the desired target, the deviations from which are penalised by public expenditures in the form of addition or subtraction of government demand from aggregate private consumption and investment. Phillips' work played an important role in presenting the concept of stability from classical control theory to the community of economists interested in finding conditions under which unwanted oscillations in an economic system, like those existing in the great depression of 1933, could be avoided.

2.3 The Early Applications of Modern Control Theory to Optimal Economic Policies

Whilst contributing to economic stabilisation policies, control theory was itself on the way towards a great advancement. Problems of regulating and controlling physical systems received considerable attention from mathematicians and control engineers following the World War II. Many ideas which had been developed in the eighteenth centuries by mathematicians, such as Euler and Lagrange, which formed the underlying mathematics now known as the calculus of variations, were used extensively to develop a systematic solution to the optimization problem. Moreover, the discovery of *Hamiltonian* formulation of variational problems in the nineteenth century proved to be very helpful to control engineers in formulating their problems.

The reorientation of the calculus of variations prompted largely by the requirements of space technology and the strong competition between the US and the ex-USSR in the field of space engineering in the 1960's. The work of Russian mathematician Pontryagin and his associates which appeared in the 1955-59 and in an English translation in 1962, known as the *maximum principle*, was a major breakthrough towards modern control theory. Bellman's contributions (1953 and 1957) to theories of multi-stage decision processes, known as the *dynamic programming*, which is based on the appealing concept of the *principle of optimality*, solved many control and system optimization problems in the late fifties and early sixties. These developments together with the advances in systems theory, particularly the *state-space representations* of systems [see Zadeh (1963)] and the related concepts of *controllability* and *observability* of systems, have established what is now known as modern control theory.

In what follows, the application of modern optimal control to optimality conditions in models of economic growth is presented in section 2.3.1. The interaction between engineering control theory and econometrics is studied is section 2.3.2 where the contributions of control engineers as well as of control engineering institutions to advance the applications of modern optimal control theory to economic policy optimisation are examined.

2.3.1 Optimality Conditions in Models of Economic Growth

The developments of theories of economic growth paved the way for economic applications of optimal control. Historically, the revival of interest in growth theories can be traced back to the period of reconstruction efforts after the Second World War as well as to the strong competition between industrialised countries to attain higher rates of economic growth. The formulation of highly aggregated models of economic growth reached its turning point in the work of Solow (1956) and Swan (1956). These models were able to generate the complete time path of each variable in the model given their initial values. The saving ratio $s(t)$ and the rate of population growth $\gamma(t)$, played important roles in the dynamics of these models since solutions to time paths of other variables were dependent on their values. By the end of 1950's it was realized that government policies can influence these parameters, thus generating different time paths for economic state variables corresponding to different values of $s(t)$ and $\gamma(t)$. The question arose as to what were the best or optimum values of $s(t)$ and $\gamma(t)$ which could lead to the desired feasible trajectories?

An optimality criterion was required to define the optimum values of parameters and the optimal trajectories. Soon it was agreed that the maximization of the integral of the utility derived from consumption per capita can be used as an optimality criterion (or objective function). Thus

$$J = \int_0^T U\{[1 - s(t)]Y(t), t\}\, dt, \tag{2.1}$$

where U is the utility derived from consumption and $Y(t)$ is income.

The equation of motion, or system dynamics, used within this framework was usually of the following structure,

$$\dot{K}(t) = s\, F[K(t)] - \delta\, K(t), \tag{2.2}$$

where $K(t)$ is the capital stock and δ is the rate of depreciation. Since output is assumed to be $Y(t) = F[K(t)]$, the objective function (2.2) can be written in terms of $K(t)$ and $s(t)$ as follows,

$$J = \int_0^T U\{[1 - s(t)]F[K(t)]\}\, dt. \tag{2.3}$$

Generalization of equation (2.1) is

$$J = \int_0^T \alpha(t) U[C(t)]\, dt, \tag{2.4}$$

where α is a bounded non-negative and continuous real discounting factor and U is a concave real function.

Studies in capital theory in growth economics, which dominated economic literature in the 1950's, was formally regarded as a problem in the calculus of variations as early as the 1930's [see Joan Robinson (1956)]. Applications of optimal control profoundly transformed capital theory to the extent that "it rechristened growth theory and has come to grips with numerous important practical and theoretical issues that previously could not even be formulated".[35]

Since equations of motion in optimal growth theory were usually expressed in terms of differential equations, the maximum principle was regarded as the

[35] See Dorfman (1969), page 817.

most efficient way of deriving the optimal conditions of time paths for economic state and control variables. In the early 1960's the community of mathematical economists were familiar with Pontryagin's maximum principle. With the well-developed literature in growth economics, the 1960's was the decade of applications of the maximum principle to theories of growth and stabilisation. Within this framework, the objectives of most research work in this period were to prove the existence of an optimal plan [Yaari (1964)], to provide economic interpretation of the maximum principle [see Dorfman (1969)] and a careful examination of optimality conditions in models of economic growth using control techniques [see, for example, Shell (1967), Burmeister and Dobell (1970) and Foley and Sidrauski (1971)]. The analytical framework was usually the maximization of a utility function subject to some constraints. For mathematical conveniences, there were only a very limited number of control variables such as government investment expenditures and the rate of interest. By applying Pontryagin's maximum principle the optimal paths for state variables were derived. For the most part, only the qualitative properties of these trajectories were analysed and no attempts at computation of optimal trajectories were made. Phase-diagram analysis was common and the planning horizon was taken to be either finite or infinite.

Bellman's dynamic programming which appeared in the 1950's, did not strongly influence the community of mathematical economists until the late 1960's. This could have been partially due to the following two factors.

i) In the 1950's mathematical economists were fully aware of Ramsey's work and *Ramsey-type* models. The necessary conditions of optimality, given by Pontryagin's maximum principle, was an ideal technique for generating the optimal paths of economic variables. However, in Ramsey-type models, optimal solutions were obtained by classical techniques of Euler-Lagrange equations. Pontryagin's maximum principle, therefore, was considered as a straightforward and more effective technique for the analysis of optimal growth problems.

ii) The main objective of most mathematical economists during this period was mathematical formulations of economic optimization problems as well as analyzing the properties of optimal paths of economic variables rather than numerical solutions in the form of computing optimal trajectories. Pontryagin's maximum principle served better towards achieving this objective as compared with Bellman's dynamic programming which was more oriented towards the computation of optimal control solutions.

The above argument might explain why the successful applications of the dynamic programming to the analysis of optimality conditions in economic behaviour appeared rather late. The two earliest interesting work on economic applicabilities of dynamic programming were Simon (1956) and Radner (1967). The former proved the *certainty equivalent control* problem in optimal growth models with a quadratic objective function and linear dynamics, while the latter successfully formulated optimal economic growth in terms of the functional equation approach in the dynamic programming.

Uncertainties in optimal growth models make the computations of optimal trajectories for non-linear models extremely complicated. Simon (1956) demonstrated, for the first time, that in the case of a quadratic objective function and a linear model with uncertainty, the determination of optimal strategies becomes very simple. In this class of problems the "uncertain" future values of variables can be replaced by their unconditional expectations, thus reducing the stochastic problem to a deterministic one.

Using a welfare function, which measured the maximum total discounted utility that can be achieved starting from a given initial state of the economy, Radner (1967) demonstrated the properties of its continuity and concavity. He studied this problem for both a finite and an infinite planning horizon and showed how the application of the dynamic programming is superior to the maximum principle with regard to the number of constraints involved. Although the basic shortcoming of the dynamic programming was its computational difficulties when applied to large non-linear models, for models of small dimensions this technique could effectively treat state and control constraints. Randner used the concept of *production correspondence* which gives, for each state of the economy, a set of alternative states to which the economy can move in the next period. This is exactly the *production possibilities* of an economy expressed in terms of the dynamic programming formulations.

Samuelson (1969) and Merton (1969) can be considered as the most successful early applications of the dynamic programming to optimum economic policy making problems. Using a welfare function with discounted utility, Samuelson (for discrete case) and Merton (for continuous case) demonstrated how the dynamic programming can be utilised in determining the optimal consumption behaviour of an individual who is facing a portfolio selection. Assuming $W(t)$ as total wealth at time t and $C(t)$ as consumption per unit of time at time t, Merton used the following objective function and the equation of motion for the continuous case,

$$J = \int_0^T e^{-\rho t} U[C(t)]\, dt, \tag{2.5}$$

$$C(t) = rW(t) - \dot{W}(t), \tag{2.6}$$

where r is the exogenously given rate of yield. Similar equations for discrete time models were developed by Samuelson, as follows,

$$J = \sum_{t=0}^{T} (1+\rho)^{-t} U(C_t), \tag{2.7}$$

and

$$C_t = W_t - \frac{W_{t+1}}{1+r}. \tag{2.8}$$

Using a recurrence functional equation, Samuelson derived the optimality condition for portfolio selections.

The subsequent research work in the dynamic programming applications to models of economic growth in this decade were mainly directed towards demonstrating that the results obtained by the maximum principle applications were equally obtainable by applying the dynamic programming technique [see, for example, Intriligator (1971)].

2.3.2 Engineering Control Theory and Econometrics

By the early 1970's econometricians had started to apply control techniques to more complex multi-state and multi-control economic systems. Soon it was fully understood that further developments in this area should go beyond the applications of the maximum principle or the dynamic programming to simple deterministic linear econometric models. The objective of most econometricians was to apply the recent advances in modern control theory, such as stochastic optimal control, optimal estimation and Kalman filters, adaptive controls and various computational algorithms such as conjugate gradient and Newton-Raphson methods, to optimal economic trajectory computations.

With their strong background in stochastic processes and estimation methods and their experiences in working with relatively large systems of simultaneous equations, econometricians were expected to contribute profoundly to the estimation and computations of optimal economic trajectories in dynamic optimization of economic systems. Fortunately, the following developments greatly assisted the community of econometricians towards these contributions.

1. Contributions of Control Engineers and Control Theorists

A number of modern control engineers and control theorists developed an interest in economic modelling, in general and econometrics, in particular. We start with Pindyck (1971). He provided a complete demonstration of the application of discrete-time Pontryagin's maximum principle in computing the optimal state and control trajectories for a small deterministic linear model of the post-Korean US economy.

Pindyck's work, originally his Ph.D thesis at M.I.T in 1971, which was then published in 1973, significantly influenced the direction of subsequent research work in econometric applications of optimal control theory. He treated economic stabilisation policies as a tracking problem in optimal control theory in which the objective of optimization was to track the desired (nominal) state and the desired control paths. The model included several basic macroeconomic state variables such as consumption, investment, GNP, interest rate, price level, wages and unemployment. The policy or control variables were the money supply, government spending and taxes. Fiscal policies were provided for through exogenous government expenditures as well as surtax and the monetary policy was realized in the money supply. By defining new state variables to replace those with lags greater than one period and adding their definitional equations to the model, he represented his model in state space format before applying Pontryagin's maximum principle to obtain optimal economic strategies.

His model, in terms of linear difference equations and in state space format, is given by

$$Y_{t+1} - Y_t = AY_t + BU_t + CZ_t, \tag{2.9}$$

with the given initial condition Y_0. Y_t is an n-dimensional state vector at time t, U_t is an m-dimensional control vector and Z_t an r-dimensional vector representing exogenous variables which are known for all t but cannot be controlled by the optimization process. A, B and C are the relevant matrices which identify the equation of motion (2.9).

Let Y_t^d and U_t^d stand for the desired or nominal state and control vectors, respectively, that we wish to track. These were assumed to be specified over the entire planning horizon. The question is to track the desired state and the desired control trajectories subject to a quadratic objective function, or cost function, stated below and also subject to a set of constraints imposed by the linear economic system given by equation (2.9). The objective function is given by

$$J = \frac{1}{2}\sum_{t=0}^{T}[(Y_t - Y_t^d)'Q(Y_t - Y_t^d) + (U_t - U_t^d)'R(U_t - U_t^d)], \qquad (2.10)$$

where Q is an $n \times n$ positive semi-definite matrix and R is an $m \times m$ positive definite matrix.[36] Matrices Q and R enable the policy-maker to penalise the deviations of state and control variables from their desired trajectories.

The problem is to find an optimal control sequence $\{U_t^{opt}, \quad t = 0, 1, \cdots, T-1\}$ which minimises the objective function (2.10) subject to the linear dynamic system equation (2.9). Both Q and R are usually diagonal matrices and their elements give the *relative costs* of deviating from the nominal paths of each state and control variables; for example, the cost of deviating from the desired unemployment relative to the cost of deviating from the desired value of inflation. Similarly, the elements of R give the relative costs of deviating from the desired paths of control variables; for instance, the cost involved in manipulating tax rates as compared to that of the money supply. Q and R can be time-varying which indicate the ranking variations of policy-makers on the relative importance of deviations over time. There are a number of problems with quadratic objective functions,[37] but

[36] R is positive definite because the optimal solution for state and control variables includes R^{-1}.

[37] The very arbitrariness of weighting matrices can be regarded as the main shortcoming of

their significance is the mathematical property known as the *certainty equivalence* discussed earlier, that linear systems with quadratic performance functions produce control laws which are linear and thus computationally tractable. Relying on this property, Pindyck did not consider the effects of additive random terms in the optimization process.

It can be shown[38] that the optimal control law for the linear system (2.9) with quadratic objective function (2.10), used by Pindyck, is as follows,

$$U_t^{opt} = -(R + B'\Phi_{t+1}B)^{-1}B'\Phi_{t+1}(I + A)Y_t^{opt} +$$
$$(R + B'\Phi_{t+1}B)^{-1}B'\Phi_{t+1}BR^{-1}B'\Theta_{t+1} - R^{-1}B'\Theta_{t+1} -$$
$$(R + B'\Phi_{t+1}B)^{-1}B'\Phi_{t+1}(BU_t^d + CZ_t) + U_t^d, \tag{2.11}$$

where Y_t^{opt} and U_t^d are the optimal values of state variables and the desired values of control variables. Φ can be obtained from the so-called *matrix Riccati equation* as follows,

$$\Phi_t = Q + (I + A)'[\Phi_{t+1} - \Phi_{t+1}B(R + B'\Phi_{t+1}B)^{-1}B'\Phi_{t+1}](I + A). \tag{2.12}$$

The values of Θ can be obtained from the following *tracking equation*,

$$\Theta_t = -(I + A)'[\Phi_{t+1} - \Phi_{t+1}B(R + B'\Phi_{t+1}B)^{-1}B'\Phi_{t+1}]BR^{-1}B'\Theta_{t+1} +$$
$$(I+A)'\Theta_{t+1} + (I+A)'[\Phi_{t+1} - \Phi_{t+1}B(R+B'\Phi_{t+1}B)^{-1}B'\Phi_{t+1}].(BU_t^d + CZ_t) - QY_t^{opt}. \tag{2.13}$$

objective functions in an economic control problem. It may be impossible to reduce the complex process of ranking economic priorities into a relatively straight-forward exercise of determining elements of weighting matrices. Moreover, there exists irreconcilable differences in opinions in political life which cannot directly be manifested in these matrices. For quadratic objective functions, however, the main shortcoming is that, these functions, as far as the penalization are concerned, cannot differentiate between the directions of deviations of optimal from desired values. Although such differentiation, in an engineering application, is not usually important, it is a decisive issue whether, for example, unemployment or inflation targets are over-or under-reached. Despite the fact that nonquadratic objective functions are better alternatives in formulating economic policies and priorities, theories and algorithms for this class of objective functions are not well-developed.

[38]See, for example, Kwakernaak and Sivan (1972).

With the boundary conditions

$$\Phi_T = Q, \tag{2.14}$$

and

$$\Theta_T = -QY_T^d, \tag{2.15}$$

the values of Φ and Θ can be computed by solving the matrix Riccati equation (2.12) and the tracking equation (2.13) backward in time.

To summarise the optimal policy computations in Pindyck's type model, we start by identifying matrices A, B and C in the equation of motion and the weighting matrices Q and R in the objective function. Once the desired values of state and control variables, Y_t^d and U_t^d, are known, the backward solutions of equations (2.12) and (2.13), the matrix Riccati and the tracking equations, together with the boundary conditions (2.14) and (2.15), will produce the values of Φ_t and Θ_t. Using the initial state of the economy, Y_0, we can compute the optimal control for period 0, i.e. U_0^{opt}, from equation (2.11). Y_1^{opt} can be obtained from equation (2.9) using U_0^{opt}. Use Y_1^{opt} in equation (2.11) to compute U_1^{opt} which, by using equation (2.9), gives Y_2^{opt}. We can continue this process until the entire sequence of optimal policies $\{U_t^{opt}, \quad t = 0, 1, \cdots, T-1\}$ are determined. Optimal state variables $\{Y_t^{opt}, \quad t = 1, 2, \cdots, T\}$ will accordingly be computed from equation (2.9).

Pindyck performed several experiments using different objective functions with different weighting matrices. Changing the elements of weighting matrices provided more insight into the trade-offs inherent in policy formulations. Such experiments demonstrated that optimal control of economic models as a tracking problem is valuable both as a tool for policy planning and as a method of analyzing the dynamic properties of economic models.

Following Pindyck, a number of researchers with engineering and mathematical backgrounds worked on similar models. The earliest examples are Paryani (1972) and Erickson and Norton (1973). The deterministic economic control problem in its general form, i.e. a non-linear dynamic economic model with a nonquadratic objective function has also been studied by a number of control engineers. More specifically, a deterministic non-linear economic control system can be modelled as

$$Y_{t+1} = f[Y_t, U_t, t].$$

The problem is to find an optimal *closed loop* policy sequence of the form

$$U^{opt}[Y_t, t], \qquad t = 0, 1, 2, \cdots, T-1,$$

which minimises the objective function

$$J = \sum_{t=0}^{T-1} \phi[Y_{t+1}, U_t, t] + \rho(Y_T).$$

In this regard, the earliest control engineers who attempted this class of problems were Norman and Norman (1973) and Athans *et al* (1975).

Contributions of control engineers in applying stochastic control theory to optimum economic policy design will be discussed in section (2.4). To complete this section, we mention that the following early contributions were quite influential in subsequent research work in this field: Upadhyay (1973) successfully applied adaptive control theory to Pindyck's model; Dyukalove (1975) analysed a number of interesting optimality features in linear dynamic economic planning models; Kendrick (1976) and Tse (1975) significantly contributed to stochastic and adaptive control of linear and non-linear economic models; and finally, Chow (1975) and Aoki (1976), the two well-known control theorists, applied a number of important concepts, techniques, and theories of modern deterministic, stochastic and adaptive optimal control to economic analysis and profoundly contributed to this field.

2. Contributions of Control Engineering Institutions

As early as the 1970's, the well-known control engineering societies such as the Institute of Electrical and Electronic Engineers *IEEE* and the International Federation of Automatic Control *IFAC*, became interested in control of economic systems and included the relevant research work in their journals and conferences. Examples of such conferences in the 1970's include: the *IEEE* Conferences on Decision and Control; Joint Automatic Control Conferences; Conferences on Dynamic Modelling and Control of National Economies; and *IEEE* Conferences on Systems, Man and Cybernetics. Although certain control engineering journals, such as *IEEE Transactions on Automatic Control*; *Automatica*; *International Journal of System Science*; *IEEE Transactions on Systems, Man and Cybernetics*; *Automation and Remote Control*, were not widely enough read by economists, there is no doubt that the great many articles of high quality published in them during this decade have made a significant contribution in generating and encouraging interests among control engineers towards optimal control of economic systems.[39]

During the 1970's, conferences organised by the National Bureau of Economic Research [see *NBER* (1972-76)], the Social Science Research Council (1972) and the USSR Academy of Sciences Central Mathematical Economics Institute (1971, 1974), have all been particularly useful in fostering a hospitable environment for co-operation between control engineers and economists. Furthermore, the publications of specialised journals such as *Economic Computations and Economic Cybernetics Studies and Research* and *Journal of Economic Dynamics and Control*, in this period, have also been enriching.

[39]For a list of 347 such work during this period, see Derakhshan(1978).

2.4 Stochastic and Adaptive Control Applications to Economic Policy Optimisation

In previous sections, we discussed the early efforts of economists and control engineers to apply deterministic control theory to optimal economic policy formulations. This section considers how stochastic and adaptive control theories were used to formulate the same problem.

2.4.1 Stochastic Control Applications

The stochastic control treatment of economic systems can be viewed in at least two different categories: economic models with uncertainties in their system's dynamics (equations of motion) and those which also contain measurement errors. Both categories might be represented by linear or non-linear systems with quadratic or nonquadratic objective functions.

Controlling a discrete-time linear system with i) uncertainties in its dynamics, ii) the presence of measurement errors and iii) a quadratic objective function can be modelled as follows. Find a policy sequence $\{U_t^{opt}, \quad t = 0, 1, 2, \cdots, T-1\}$ to minimise

$$J = E\,[(Y_T - Y_T^d)' S (Y_T - Y_T^d) + \sum_{t=0}^{T-1} (Y_t - Y_t^d)' Q_t (Y_t - Y_t^d) + \sum_{t=0}^{T-1} (U_t - U_t^d)' R_t (U_t - U_t^d)], \tag{2.16}$$

subject to the linear stochastic model presented in the following state space format

$$Y_{t+1} = AY_t + BU_t + \xi_t, \tag{2.17}$$

and the measurement subsystem,

$$Z_t = DY_t + \omega_t, \tag{2.18}$$

where Z_t is j-dimensional observation vector, ξ_t and ω_t are n- and j-dimensional system and measurement noise processes with zero mean and covariance matrices

Ξ and Ω, respectively. All other terms are defined earlier. In this class of problems, it is assumed that the state of the economy is observed through a stochastic measurement subsystem. There exists two sources of uncertainties: additive noise ξ_t in the state space representation of the economic model and additive measurement noise ω_t in the measurement sub-system. We are now faced with a more complicated problem, as compared with the deterministic case, but it is a more realistic one.

The certainty equivalence theorem of Simon (1956) and Theil (1957, 1964) can easily be applied to the class of linear systems represented by equations (2.17) and (2.18) and quadratic objective function (2.16). Among the earliest attempts to solve this problem are Chow (1972), Brito and Hester (1974), Bray (1974) and Phelps and Taylor (1975). A systematic analysis of non-linear economic models with additive noise and quadratic performance measure is discussed for the first time by Garbade (1975).

Contributions of Westcott and Wall (1976) to stochastic policy optimization of economic systems, published in an engineering journal called *Automatica*, significantly influenced the subsequent direction of research work in economic applications of stochastic control theory. They used a linear stochastic control model together with a quadratic performance measure and a Gaussian distribution of disturbances. The model was designed to obtain optimal economic strategies for the four major problems confronting the UK policy-makers, i.e. the behaviour of unemployment, price and wage inflation, the balance of payments, and economic growth. There were thirteen behavioural equations estimated using quarterly data over the period 1955-1973, specifying the behaviour of unemployment, employment, private investment, stock building, private consumption, consumer price index at factor cost, wages, profits, real exports, real imports, export price index, import price index and factor cost adjustments. There were thirteen definitional equations in the model. The parameters were estimated by a dynamic generalization of simultaneous multivariate maximum likelihood estimators in conjunction with residual correlation diagnosis.

The estimated model is then converted to an equivalent state space format to apply the technique of stochastic optimal control. A minimal realization procedure, developed by Prescott and Wall (1973), was used to obtain the minimal state space dimension. The standard approach of making one state for each lagged variables, as has been used by Pindyck's control model discussed in equation (2.9)], is avoided here, otherwise a model of 46 state variables as compared to only 29 variables would have been obtained. The behavioural equations were expressed in terms of growth rates which is equivalent to the first difference in natural logarithms.

The model in its polynomial structural form and in their own notations is as follows.

$$\begin{aligned} A_0 Y_k = \sum_1^p A_i L^i Y_k + \sum_0^\sigma B_i L^i U_k + \sum_0^i C_i L^i e_k + \\ \sum_0^s \Delta_i L^i Z_k + \sum_0^t \Gamma_i L^i \omega_k, \end{aligned} \tag{2.19}$$

and

$$Z_k = J_1 Y_k + J_2 U_k + J_3 e_k. \tag{2.20}$$

L is the lag operator where $L^j X_k = X_{k-j}$ and Y_k is an l-vector of endogenous variables, U_k is an m-vector of instruments or control variables, e_k is an m_1-vector of exogenous variables, Z_k is an m_2-vector of definitional variables or identities, ω_k is an l-vector of residual variables or stochastic disturbances and the matrices A_i, B_i, C_i, Δ_i and Γ_i are all real. The disturbance vector ω_k is assumed to have a Gaussian distribution. It is also assumed that the objective function can sufficiently be approximated by the expected value of a quadratic function in terms of the deviations of state and control variables from their desired trajectories. In their notation, the objective function is given by

$$W = \frac{1}{2}E\{\delta Y_N' L_1 \delta Y_N + \delta U_N' L_2 \delta U_N + \sum_1^{N-1}[\delta Y_k' A \delta Y_k + 2\delta U_k' C \delta Y_k + \delta U_k' B U_k]\}, \quad (2.21)$$

where L_2 and B are $m \times m$ symmetric positive definite matrices and L_1 and A are $l \times l$ symmetric positive semi-definite matrices. The deviations $\{\delta Y_k; \quad 1 \leq k \leq N\}$ and $\{\delta U_k; \quad 1 \leq k \leq N\}$ are taken with respect to the nominal state and nominal control trajectories. The problem is to find a sequence of control variables to minimise the objective function, W, subject to the linear economic model [equations (2.19) and (2.20)].

Optimal policy experiments were carried out for 20 quarters. The main objective was to demonstrate which instruments were most effective in achieving a given target or policy goal. For example, should only the foreign exchange rate be used to balance trade, or should fiscal policies be applied instead or as well. The equivalently important problem was how to coordinate instruments in order to achieve simultaneously a given combination of policy goals, which might involve problems in optimization under conflicting objectives. With regard to the first problem, the elements of weighting matrices in the objective function can be manipulated to identify which control variables are the most suitable in reaching a specific economic objective. Westcott and Wall's experiments demonstrated that fiscal policies, such as public investment and social expenditures, were active instruments in both balancing the trade and stimulating growth.

2.4.2 Adaptive Control Applications

A new extension within the framework of optimal control of stochastic dynamic economic models has been achieved by introducing uncertainties in the system's parameters. This branch of stochastic control is also known as adaptive, self-organizing, self-optimising, self-regulating and learning system. In this case, stochastic state space representation of systems, i.e. equation (2.17), will become

$$Y_{t+1} = A(\psi, t)Y_t + B(\psi, t)U_t + \xi_t, \tag{2.22}$$

where $A(\cdots)$ and $B(\cdots)$ are $n \times n$ and $n \times m$ state transition and driving matrices, respectively. ψ is a fixed unknown l-dimensional vector taking values in parameter space Ψ. ξ_t is an n-dimensional system dynamic disturbance vector. We can also assume that some or all of the endogenous variables in the economic model are not available for exact measurement. A sub-system for measurements in which the observations are assumed to be linear functions of endogenous variables with additive random disturbances will follow,

$$Z_t = C(\psi, t)Y_t + \omega_t, \tag{2.23}$$

where Z_t is a j-dimensional observation vector and ω_t is a j-dimensional measurement noise process. ξ_t and ω_t are assumed to be independent Gaussian noise with zero means and covariance matrices Ξ and Ω, respectively. The objective function is assumed to be quadratic and additive in state and control as expressed by equation (2.16). The feedback control U_t, at any time t, is assumed to be a function of the observation, i.e. the state measurement history. If we define

$$\eta_t = \{Z_0, Z_1, \cdots, Z_t\},$$

then

$$U_t = h(\eta_t), \quad t = 0, 1, \cdots, T-1. \tag{2.24}$$

The adaptive control of a linear economic model with unknown parameters is to find the optimal economic decision sequence $\{U_t, \quad t = 0, 1, 2, \cdots, T-1\}$ which yields the minimum value to the objective function defined by equation (2.16) subject to the economic model and the measurement sub-system specified by equations (2.22) and (2.23), respectively. In this class of problems, the accuracy of the estimation is a function of the control action while the quality of control will depend upon the degree of accuracy by which the economic model is estimated. The controller must, therefore, compromise between estimation and control. This problem is usually referred to as the *dual control problem.* The uncertain parameters are usually regarded as additional state variables. Such treatment of parametric uncertainties translates most linear econometric models into essentially a problem in non-linear stochastic control theory and thus takes the econometricians into the realm of non-linear estimation theory. The Bayesian approach has the potential to solve this class of problems where there exists *a priori* knowledge about probabilities of the unknown parameters.

Amongst the most successful early applications of adaptive control to economic policy design problems are Chow (1973), Upadhyay (1973), Kendrick and Majors (1974) and Turnovsky (1975). Upadhyay (1973) applied the method of adaptive control to the recursive linear difference equation model of the US economy developed by Pindyck. As discussed in section (2.3.2), the Pindyck model was a deterministic optimal control of a linear system with a quadratic objective function. Upadhyay extended Pindyck's model by making the parameters of the model as random variables with assumed statistics. He formulated the simultaneous estimation and control of Pindyck's model, and showed that the unknown parameters in the model can be identified while simultaneously controlling the economy. His results indicate the advantages of applying adaptive control techniques to economic modelling and control. Using the average value of the objective function as a measure of comparison, he demonstrated that the adaptive control scheme is "better", i.e. yields smaller value of the objective function as compared to the optimal deterministic control approach.

The increasing familiarity of economists with modern control theory and their improved cooperations with control theorists in the 1970's resulted in a state of self-confidence in the literature on economic control applications. However, having experienced the so-called *rational expectations revolution*, the community of economists interested in control theory were faced with the following question in the early 1980's. What exactly has control theory offered to economists towards improving the quality of optimal decision making processes? To answer this question, we examine in sections 2.6 and 2.7 the impact of rational expectations, the Lucas critique and the time-inconsistency on the standard applications of optimal control theory to dynamic choice in economic models. But let us first examine, in section 2.5, the recent developments in macro-econometric modelling in the context of dynamic optimisation. This provides a background to sections 2.6 and 2.7.

2.5 The Relationship Between Theory and Observation: A Critical Analysis of the Recent Developments in Macro-econometric Modelling and the Role of Dynamic Optimisation

The question we posed in Chapter One (section 1.3.3) that what has been gained by the Cowles Commission's efforts in the 1930s on unifying economic theory, mathematical methods and statistics has received wide attention during the past 10 years especially with regard to macro-econometric modelling and advances in dynamic optimisation of behaviour of a representative economic agent. Pesaran and Smith, among others, have systematically addressed this issue since 1985 within the framework of the relationship between theory and observation (see Pesaran and Smith, 1985a, 1985b, 1988, 1992a, 1992b, 1995a, 1995b). More recently, an interesting debate on data dependence versus theory dependence approaches to model building in applied econometrics has appeared in the *Economic Journal* (November 1995) in which Danny Quah, Allan Gregory, Gregor Smith, Martin Eichenbaum, David Hendry and Michael Wickens have participated. In what follows, I first examine the core of the argument in this field. The analysis of the origin of the gap between data-instigated and theory-based econometric models are then discussed and the role of dynamic optimising models together with a critical examination of the attempts to bridge this gap are presented. A critical analysis of Pesaran and Smith's contribution, i.e. the importance of *economic theory* in tightening the link between theory and empirical work, is also included in this section.

2.5.1 Responses to the Cowles Commission's Traditional Strategy

From an historical point of view, the literature on the interaction between theory and observation in applied econometric models can best be analysed by dividing the subject into the traditional Cowles Commission's approach and the reactions to it. The latter have mainly appeared in the two broad categories of the atheoretical or data-based vector autoregression (VAR) and the astatistical or theory-based calibrated models. Developments in the economic modelling aimed at combining statistical and theoretical information is another dimension of research work to be considered.

The dynamic optimisation approach which provides a strong link between neoclassical optimisation theory and econometric models has recently come to play a much more prominent role in applied econometrics. This approach is extremely important since it requires all the behavioural equations in an econometric model to be obtained directly by the solution of intertemporal optimisation problem facing an economic agent. This approach is discussed in the next section where we present a critical analysis of the different strands of research work which seek to bridge the gap between theory and observation in applied econometrics.

Keynes' strong criticisms on the first macro-econometric model, developed by

Tinbergen (1939) to test the business cycle theories, formally initiated the debate on the validity of econometric models in unifying theories and observation in economics (see Pesaran and Smith, 1985b). The role of theory, data, endogeneity, linearity, dynamics and structural stability were among the issues which raised. The Cowles Commission's major task to provide a solution to these problems formed a solid basis for the rapid expansion of macro-econometric model building in the 1950s and 1960s. This traditional approach, based on the Haavelmo's probability foundation, (Haavelmo, 1944), which remained unrivalled until the 1970s (see Morgan, 1990), is characterised by the dichotomy between theoretical and empirical activities in which mathematical economists and econometricians provided the model and parameter estimates, respectively. The role of the theory was, according to Tinbergen (1939), confined to "identifying the list of relevant variables to be included in the analysis, with possibly the plausible signs of their coefficients". In this framework, regression analysis was the main focus of econometricians with the prime objective of obtaining conditional predictions.

By allowing for different explanatory variables, choosing different functional forms for the behavioural equations, introducing various time-lags structures to account for dynamic adjustments (such as Koyck and Almon), using dummy variables for certain unobservable explanatory variables and introducing different assumptions about the stochastic terms in structural equations, the regression analysis provided a state of flexibility which made the theory almost unfalsifiable. With much emphasis on estimators (in contrast to testing procedures, see Qin, 1991) the Cowles Commission's Programme in Economics soon produced a variety of estimators designed to estimate a system of simultaneous equations. These estimators included instrumental variables, limited information maximum likelihood, full information maximum likelihood, two- and three-stage least squares and k-class estimators.

Koopmans' method of identification provided an interesting solution to the Cowles Commission's problem of unification of theory and observation (see Koopmans, 1950). It is well-known that the rank and order conditions for identification of a single equation in a system of simultaneous equations are based on the *a priori* information supplied by economic theory about the linear or non-linear restrictions among structural parameters. In other words, without such *a priori* theoretical information many different structural forms can become consistent with a single reduced form (which expresses an endogenous variable in terms of exogenous variables and thus can be estimated by the OLS method). It follows, therefore, that without theoretical restrictions structural equations may become observationally equivalent. Similarly, one can argue that the identification of a single equation in a system of simultaneous equations is equivalent to the problem of whether the estimates of structural parameters can be obtained from the estimated parameters in the reduced form. If more than one estimated structural parameters is possible the equation is said to be over-identified for which the method of 2SLS was developed yielding unique estimated values of structural parameters. This clearly signifies how economic theory could provide a basis for further developments in econometrics.

Despite the outstanding success of the Cowles Commission's traditional econometric practice in estimating Keynesian and neo-classical models and its popularity amongst the community of business and official policy-makers,[40] it received a number of different shocks during the 1970s. The increasing evidence that the traditional models did not adequately represent the data led to a growing theoretical emphasis on dynamic specification and model selection in econometrics. This provides an example of how the data inconsistency could enhance further developments in econometric analysis. Moreover, theoretical developments in ra-

[40]See Bokin, Klein and Marwah (1991) for an historical account of this success.

tional expectations, time-inconsistency and dynamic optimisation persuaded the theorists to believe that the traditional econometric models cannot successfully represent theory. This led to the Lucas critique. The response of mathematical economists and econometricians were, respectively, to give much greater priority to representing theory at the expense of statistical analysis and to representing data at the expense of theory. The theory-based calibrated models and the data-based vector autoregressions discussed below are the two well-known manifestations of this response.

The astatistical approach in the theory-based calibrated models can best be seen in the *Computable General Equilibrium Models* (CGMs). "In calibrated models, the parameters are chosen on the basis of introspection, surveys of previous econometric results or set to match certain features of the data; frequently taken to be the unconditional moments" (see Pesaran and Smith, 1995b, p. 7). As the title implies, the objective of computable general equilibrium models is to develop a computable solution to the Walrasian general equilibrium models which mainly relied on calibration methods rather than using econometric techniques.[41]

Placing greater emphasis on theory-based economic models is usually accompanied by greater reliance on *pragmatic empirical* work. According to Summers (1991) a pragmatic empirical work is an approach which is based on *stylised facts* and is easy to understand and simple to use. He refers to the fact that many major contributions to economic modelling, such as Friedman's work on the consumption function or the work of Solow and Dennison on growth theory, constitute examples of pragmatic research in which empirical regularities can easily be understood without using any formal econometric techniques. Summers concludes

[41]Dervis, De Melo and Robinson (1982) and Shoven and Whalley (1992) provide a review of the CGMs. It should be noted that CGMs have their roots in the classical contributions of Leontief (1941) and of Leif Johansen (1960). The latter developed a multi-sectoral growth model for Norway.

that "formal econometric work, where elaborate technique is used to apply theory to data or isolate the direction of causal relationships when they are not obvious *a priori*, virtually always fails" (p. 129).

The data-instigated or atheoretical approach of Sims (1980), known as the data-based vector autoregressions (VAR), is another response to the failure of the traditional approach of the Cowles Commission. As we discussed earlier, the regression analysis (single or multivariate models), which is called by Hylleberg and Paldam (1991) the *traditional strategy* of marrying theory and observations, was an effective method in empirical economic modelling until the 1970s because the functions of the theory involved were to identify the list of relevant explanatory variables (and possibly the plausible signs of their coefficients) and to provide some linear or non-linear parametric restrictions. Although not many test procedures were included in the traditional strategy (for example, the possibility of pre-testing and specification searches discussed by Leamer, 1978), the comparison of the conditional predictions provided by the model with the actual realisations could have allowed an informal testing of the plausibility of the statistical methods of estimation. The rise of the *New Classical Macroeconomics*, (see Lucas and Sargent, 1981), with emphasis on the rational expectations and general equilibrium modelling, as well as the partial failure of Keynesian macro-econometric models in predicting the stagflation of the 1970s, provided heavy criticisms of the traditional strategy. It was soon realised that factors such as unobservable theoretical variables, inaccurate functional forms and appropriate adjustment processes render all econometric models significant misspecification errors. This, together with the recognition of the incompleteness of economic theories, paved the way for a data-instigated atheoretical approach to empirical economic research work.

The univariate ARIMA model, which represents a variable in terms of its lagged values of a moving average of lagged disturbances, was a successful earlier attempt in this new approach. Despite using no information from economic theory and thus being atheoretical, economic theory could at some occasion provide certain restrictions on these models (for example, speculative asset prices should be a random walk in efficient markets thus, ARIMA(0,1,0) follows). The response of econometricians to the relative success of the ARIMA models in out-performing traditional regression models in forecasting was twofold: giving more emphasis on model specification, diagnostic and misspecification tests; and placing much greater priority in the incorporation of the information available in ARIMA models than in the traditional econometric models. In this regard, the LSE tradition associated with Sargan (1964) and Hendry (1982, 1987) known as the *Error Correction Models* (ECMs) and the VAR approach, discussed earlier, are the two major research strategies resulting from the latter response. Let us now examine how these two approaches have contributed to bridging the gap between theory and observation in applied econometric models.

Hendry's ECMs[42] is based on a general autoregressive distributed lag model in which an endogenous variable is explained by its own lags and current and lagged exogenous variables. After testing the estimated model for its *statistical adequacy*, the model will then, upon using economic theories, become more simplified by re-parametrisation and restrictions that reduce the number of estimated coefficients. This yields a single equation in which the changes in dependent variable are expressed by changes in the explanatory variables and the lagged dependent and independent variables. Evidently, only on two occasions is economic theory permitted to play its role in this approach: at the beginning when the choice of variables is being made and at the final stage with the re-parametrisation and imposition of restrictions. This completely violates the traditional strategy in

[42]See Alogoskoufis and Smith (1991) for a critical and detailed examination of this approach.

econometric modelling based on the dichotomy between formulating behavioural equations based on economic theories and estimating its parameters by using appropriate statistical methods of estimation. Spanos clearly elaborates this point in his celebrated textbook (1986, pp. 670-671) as follows: "Econometric modelling is viewed not as the estimation of theoretical relationships nor as a procedure for establishing the 'trueness' of economic theories, but as endeavour to understand observable economic phenomena using observed data in conjunction with some underlying theory in the context of a statistical framework". In Hendry's method, econometric modelling is, therefore, nothing more than characterisation of data properties in parametric relationships based on an observed sample. The role of economic theory is, therefore, marginalised to provide an economic interpretation of such relationships based on the specification of their long-run properties.

In contrast to Hendry's approach, which is a single equation structure, VAR is a multivariate method. All variables are treated as endogenous being explained by the lagged values of themselves and other endogenous variables. There are no restrictions imposed by economic theories on parameters. It is easy to see that VAR is structurally a reduced form since endogenous variables are modelled as a function of predetermined variables. Economic theories, therefore, cannot produce any economic interpretation of coefficients since these coefficients are usually a combination of structural parameters. It is well-known that except for recursive systems, economic theories can only provide explanations for structural parameters.[43] The correspondence of VAR to reduced form has led Monfort and Rabemananjara (1990) to assume the existence of a structural form for any given VAR. The question arises as to what are the sequential restrictions necessary to specify a corresponding structural form. These restrictions include the predeterminateness of some variables, exogeneity and non-causality conditions.

[43]For a critical analysis of such an environment of atheoretical econometrics see, Cooley and Le Roy (1985).

Having briefly discussed the responses to the traditional strategy in model building, we now turn to examine the origin of the gap between the data-instigated and theory-based approaches to applied econometric analysis. The role of dynamic optimising models in this debate and the various attempts to tighten the link between theory and evidence in macro-econometric models are also critically appraised.

2.5.2 Data-instigated vs Theory-based Econometric Models: The Origin of the Gap, the Role of Dynamic Optimising Models and a Critical Analysis of the Attempts to Bridge it

"Year after year economic theorists continue to produce scores of mathematical models and to explore in great detail their formal properties; and the econometricians fit algebraic functions of all possible shapes to essentially the same sets of data without being able to advance, in any perceptible way, a systematic understanding of the structure and the operations of a real economic system". This is the view expressed by Wassily Leontief, the Nobel Laureate in economics, on the state of partnership between mathematical economics and econometrics (see his foreword to Alfred Eichner, 1983, pp. *x*-*xi*). He continues to examine the origin of and the remedy for, such an unfortunate failure by saying that "That state is likely to be maintained as long as tenured members of leading economics departments continue to exercise tight control over the training, promotion and research activities of their younger faculty members and, by means of peer review, of the senior members as well". While I would agree with the structure of Leontief's broad conclusion, there are reasons to believe that his analysis does not advance, in any perceptible way, the understanding of the nature and origin of the existing gap between mathematical economics and econometrics. Equally important, his analysis does not contribute towards tightening the links between them.

Our starting point is the recognition of the fact that the unsuccessful partnership between mathematical economics and econometrics in the 1970s has given impetus for advancing the machinery of econometric analysis. This, in turn, has improved the understanding of the structure and operational properties of the real world economic life. In other words, this unsuccessful partnership has not been unproductive. Let us recapitulate the main merits and shortcomings of the different strands of developments resulting from the response to the relative failure of traditional strategy of econometric modelling before examining the place of optimising models in this debate.

We discussed in the previous section that the predictive failure of the Cowles Commission's programme in macroeconomic trends (for example, the stagflation of the 1970s) led to a substantial development in the univariate time-series analysis. In this regard, the work of Sims (1980) is a clear example. There have also been attempts to integrate data-instigated models and the traditional econometric analysis. The expanding literature on testing for unit roots, stochastic trends and the cointegration approach pioneered by Granger are the most significant advances in this field.[44]

It should be noted, however, that the vector autoregressions and the cointegration theory have basically remained data-instigated models. While it is true that the rational expectations and optimisation methods can impose some cross-equation restrictions on vector autoregression,[45] there are not sufficient reasons to believe that the VAR is not an atheoretical approach. The *definition* of economic theory, of course, plays an important role in this judgement. Without going into the methodological issues about the definition of an economic theory, we can say, with no loss of generality, that on the basis of an economic theory being a

[44]See, for example, the collection of papers in Engle and Granger (1991), Phillips (1991) and Phillips and Hansen (1990).

[45]See, for example, Hansen and Sargent, 1991.

description of causal mechanisms,[46] the vector autoregression is basically atheoretical. The same argument applies to the cointegration approach. While it is true that the cointegration approach considers the relationships between economic variables, it provides only a long-run relationship without identifying the causal direction. Again, with regard to the above mentioned definition of economic theory, the cointegration approach will remain atheoretical too.

A similar argument applies to Hendry's Error Correction Models. Although this method is less atheoretical as compared with the VAR and cointegration models, it suffers from the single equation approach (and thus from the simultaneity character of the traditional approach) as well as from the absence of a theoretical role in the specification of the time-lags and dynamic adjustments. The status of theory is thus reduced to identifying the long-term relationships between economic variables. This role can also be debatable.[47] However, within the framework of the impact of observation on theory, the Summers' *pragmatic empirical* work, based on the stylised facts rather than the elaborate formal econometric models discussed in the previous section, is the most outstanding contribution to the theory-based calibrated economic models.[48]

The significant step towards the unification of theory and observation is the dynamic optimisation approach which provides a direct link between neo-classical optimisation theory and applied econometric models. This approach, which is developed as a response to the rational expectations hypothesis and the Lucas critique, gives more emphasis to the role of economic theory in econometric modelling. All behavioural equations, in this approach, are obtained directly as solutions to the dynamic optimisation problem facing an economic agent.

[46] See, for example, Miller (1987).
[47] See Alogoskoufis and Ron Smith (1991).
[48] For a critique of the calibrated approach, see Anderson (1991).

Some major problems with dynamic optimisation approach as a bridging mechanism to link theory and measurement in econometric models can be summarised. *i*) Most of the optimising problems in the literature are based on a *representative agent* with very restrictive assumptions about preferences, technologies and the information set, such as the information homogeneity to approximate the information heterogeneity of agents discussed in Pesaran (1986 and 1990 chapter 4). Arrow (1986) has pointed out the importance of the assumptions of information heterogeneity and decentralised markets where individual differences are prime motives for trade. *ii*) Partly because of the above restrictive assumptions, some important problem, such as the choice of functional forms or the sector disaggregation, cannot successfully be accommodated in a dynamic optimisation framework. *iii*) As Sargent (1987) remarks, dynamic optimisation models based on a representative agent suffer from the lack of simultaneous interdependencies of decisions made by different agents in the context of a general equilibrium model.

We sketched out a critical examination of different efforts towards bridging the gap between mathematical modelling in theoretical economics and applied econometrics. None of the work cited above provides systematic analysis of the nature and origin of this gap. This introduces a serious theoretical shortcoming towards tightening the link between economic theory and statistical estimation. To clarify this point, let us begin with a critical examination of the research strategy of Smith and Pesaran for bridging the gap between theory and evidence.

A Critical Examination of Smith and Pesaran on the Interplay of Theory and Observation

Smith and Pesaran have systematically refined their argument on the interplay of theory and observation since 1985 and have only recently (1992 and 1995) formulated the specific structure of how economic theories can improve the performance of applied econometric models. Their earlier work implied only that tightening the link between economic theories and econometric models necessarily required a more serious contributions of observations to economic theorisation as well as contributions of economic theory to measurements: "As econometricians, not only is it important that we adapt our econometric techniques to the empirical analysis of the economic theory given to us by the theorist, but we need also to constantly search for new ways of making economic theories more suitable for econometric analysis, always bearing in mind the dangers of allowing theory to become a straitjacket" (Pesaran, 1988, pp. 338-339). In other words, the message for the role of the theory in applied econometric models was to select and adapt a theory in such a way as to make it more appropriate for applications. As Phillips (1988, p. 351) has remarked "[Pesaran's] message is, of course, quite an old one. It is strongly evident in the writing and work of early researchers like Frisch (1932) and Stone (1951) and recently it has been forcefully restated elsewhere by Hendry and Wallis (1984) and by Pesaran (1986) himself".

It is very hard to challenge Smith and Pesaran's research strategy theoretically since it is expressed in the most general terms. A question of prime practical importance is how an econometrician can balance between the too many restrictions derived from theory-based models and at the same time place heavy reliance on atheoretical data-oriented methods. As Mankiw (1989, p. 89) comments "Yet like all optimising agents, scientists face trade-offs. One theory may be more 'beautiful' while another may be easier to reconcile with observation". Pesaran himself admits this problem when he writes (1988, p. 338) that "This leaves us in an uneasy middle-of-the-road type situation". He tries to sort out this problem by making the role of theory in applied econometric models a function of the following factors (*ibid*, p. 338): 1) the objective of the exercise; 2) the nature of the problem at hand; 3) the degree to which the economic theory and the measurement model are integrated; and 4) the fashion and trends in the profession. Evidently, these facts cannot offer an empiricist engaged in an actual econometric model a practical answer to the aforementioned trade-off between theory and observations in econometric models.

The recent work of Smith and Pesaran offers a more specific treatment of the role of economic theory in applied work. They regard the application of dynamic optimisation methods in econometric models as an outstanding achievement in advancing the theory-based econometric models. They, however, admit that the very restrictive assumptions inherent in dynamic optimisation, together with the fact that the behaviour of a representative agent lies at the root of the optimisation approach, can, in principle, make economic theory a straitjacket in econometric models. Within this context, they have elaborated their earlier general and non-specific position on the importance of economic theory in the interplay of theory and observation as follows: "How can we devise a procedure that incorporates the precision of the modern dynamic stochastic theory and the flexibility of the traditional approach? The aim would be to develop a general econometric framework which enjoys the precision of the dynamic optimisation approach but does not suffer from its formal stricture when applied literally to economic problems" (Pesaran and Smith, 1992, p. 15). They have tried to achieve this objective by computing the shadow prices. Let us now examine how well they have succeeded.

It is well-known that shadow prices (Lagrange multipliers or costate vectors) arise in the optimisation problems which are subject to constraints. Shadow prices are usually related to prices of goods and services in the markets which do not exist. This explains why shadow prices are unobservable, which in turn renders serious computational problems for the estimation of non-linear dynamic optimisation. The situation will not be pressing if the theory can be cast in (or become approximated by) LQ form, i.e. linear dynamics with a quadratic objective function, since analytical solutions exist for its optimal policy sequence. However, quadratic objective functions usually imply excessive restrictions on economic theories such as asymmetric adjustment costs, irreversibilities and non-negativity of output prices or factor inputs. Such restrictions endanger economic theories of becoming a straitjacket. Obtaining conditional predictions for dynamic optimisation of non-LQ problems is, therefore, a difficult problem in applied econometric models.

Smith and Pesaran's main contribution is to replace the unobservable shadow prices (or expected shadow prices in the stochastic case) by functions (preferably linear or other tractable functions) of the variables that determine them. The resulting dynamic optimisation problem can be solved for conditional predictions. Upon confronting conditional predictions with observation the relative validity of the model can be inferred. The crucial role of economic theory in the unification of theory and observation, so much advocated by Pesaran and Smith, can basically be summarised as the replacement of the unobservable shadow prices by a functional form from observable variables. This allows economic theory to work outside the framework of dynamic optimisation not only to relax the computational aspects of estimation for prediction but to permit other relevant economic variables to enter the optimisation problem through their influence on the shadow prices.

To illustrate their method, Smith and Pesaran have considered the case of investment in which the optimisation problem gives rise to a Lagrange multiplier which is the shadow price of capital. They have solved the problem by using the assumption that a complete second-hand market for capital goods does not exist (Pesaran and Smith (1992), appendix, pp.21-23). Evidently, Bellman's dynamic programming, which recursively solves the functional equation resulting from recursive formulation of the intertemporal objective function, is not appropriate for solving such dynamic optimisation problem because it does not generate the shadow prices. However, Pontryagin's maximum principle, which provides an explicit treatment of shadow prices, linking the values of decision variables over time, can effectively handle this problem. More recently, Pesaran and Smith (1995) have used this method to obtain a substitute for the unobservable shadow prices by their determinants arising in dynamic optimisation of liquidity constrained consumer behaviour as well as in the optimisation of oil exploration and development.

Undoubtedly, Smith and Pesaran have taken a major step in solving non-linear dynamic optimisation problems. The way they have presented empirical evidence in terms of shadow prices rather than as parameters of conditional distribution of observable variables is a significant progress towards strengthening the link between mathematical economics and econometrics. Their method makes provision for some important empirical observations (such as liquidity constraints) to appear in theoretical terms which can be more apprehended by theorists. While admitting that Pesaran and Smith have provided a solution to a long-standing serious problem in estimation of optimal econometric models they have not justified the use of optimising models in building a bridge between theory and observation. Despite the fact that dynamic optimisation models offer a substantial alternative within the framework of theory-based models, the assumption of representative agent, among other serious restrictive assumptions discussed earlier, is a strong impediment in bridging the gap between theory and observation in applied econometric models.

2.5.3 Speculations on the Future Course of Developments

While the preceding discussion signifies the existing gap between theorists and econometricians, the attempts to bridge it have been conducive to advancing the state of knowledge in applied econometric models and econometric techniques. This, at the same time, has promoted our understanding of the depth of the problem. In other words, although the gap exists, we now know more about it and we are thus encouraged to learn more. A question of theoretical interest is whether the unification of theory and observation, advocated so much by the Cowles Commission Programme and the Founders of the Econometric Society, can ever be viable?[49]

At the root of the argument lies the fact that economic conceptualisations and economic observations (or measurements) are not usually mutually independent. For example, the concepts of price and output could not have originated without concomitant developments of their associated quantification in the sphere of observation and measurement. The gap started to emerge with the process of economic theorisation in which simple factual economic concepts were put in hypothetical relations in order to explain a given economic observation or to predict a course of development. The method of abstraction discussed in Chapter One has always been central in erecting the conceptual edifice of economic theorisation. The capacity of economic theory to produce unifying insights into the functioning of an economic system depends on how the theory has been abstracted from the complete mass of details called the economic reality. A theory may not necessarily be a coherent set of assertions about actual economy. However, this may not be true for theories involved in applied econometric models: the higher the degree of applicability of a theory the larger would be the extent of its assertions on observations. Theoretical concepts in highly applicable models, such as applied micro-econometric models in finance, are more closely linked with the corresponding observations and are, therefore, available to more precise measurements. In these models, the gap between theory and observation can become minimal in

[49]The objectives of the Econometric Society and the Cowles Commission are discussed in Chapter One, section 1.3.2.

the process of model building and testing since the theory will quickly adjust itself to observation. In a similar fashion, one can conclude that the adjustment process for highly aggregated macro-econometric models is very slow mainly due to the weak correspondence between their theoretical concepts and the associated observation. In these macro-econometric models the gap between theory and observation is likely to remain substantially wide.

Confrontation of theory's predictions with evidence can further support the above argument. It is known that in models of partial equilibrium, predictions are made conditional on auxiliary assumptions which are an integral part of designing an empirical econometric model. Statistical inference on conditional predictions cannot, therefore, be made by reference to the actual data which does not conceptually correspond exactly to conditional predictions. This suggests that testing the predictive property of econometric models is not basically possible and thus a careful analysis of the gap between theory and observation cannot successfully be carried out.

The importance of economic theory in applied econometric analysis has a long history. Koopmans (1947), representing the emerging econometric approach of the Cowles Commission, presents the first substantial analysis on the subject. In a critical review on the data-instigated method of Burns and Mitchell (expressed in their book entitled *Measuring Business Cycles*, 1946) he writes that "Fuller utilisation of the concepts and hypotheses of economic theory *as a part of the processes of observation and measurement* promises to be a shorter road, perhaps even the only possible road, to the understanding of cyclical fluctuations". Almost after 50 years, the work of Pesaran and Smith (1995), relating to the importance of economic theory in applied econometric models, does not extend beyond measuring the shadow prices in terms of observable economic concepts in the domain of dynamic optimisation of economic policy design. The major problem of how the applied econometric models based on non-linear dynamic optimisation can become integrated with a measurement system is still open. In this regard, applied econometrics awaits a substantial and significant contribution towards accommodating economic theories in constructing empirical models.

Another important point which has not yet received enough attention is the role which has been played by statistical analysis in the debate on the gap between theory and observation in applied economics. Since 1932, the Cowles Commission has mainly emphasised the development of different methods to estimate *population statistics* from *samples.* This has produced a rich literature on econometric methods now known as the traditional strategy. As Vining (1949) in his renowned review of the *Koopmans on the Choice of Variables to be Studied and of Methods of Measurement* says the excessive emphasis of modern statistics, upon certain types of problems stressed by Kendall, has been most influential in creating this situation. Kendall (1943) remarks that "the estimation of properties of a population from a sample is the most important practical problem in statistics and is long likely to continue so". However, Yule (1943) has denied this proposition by saying that "It never was, in my opinion, the most important practical problem in statistics ... The initial problem of the statistician is simply the description of the data presented; to tell us what the data themselves show. To this initial problem the function of sampling theory is in general entirely secondary or ancillary ... More recently methods, with few exception (time-series in economics, factor-methods in psychology), have been almost neglected, while there has been a completely lop-sided development -almost a malignant growth- of sampling theory. I hope there may be a swing back towards the study of method proper and as methods only develop in connection with practical problems, that means a swing back to more practical work and less pure theory ...".

Undoubtedly, Kendall's emphasis on *sampling theory* and *hypothesis testing* in applied statistics played a prominent role in the Cowles Commission's approach to econometric models. However, it took almost 40 years for economists to take the Yule's recommendation seriously: economic time-series analysis was revived in the 1980's. Yet the literature on applied econometric modelling awaits a major break-through to combine time-series and economic theories.

What can be done until then? At one extreme is the argument that economic modelling, as a precise academic exercise, should not be taken seriously in building real world's economic models. "All empirical methods are simply rough guides to understanding -trying to take them as more seems misguided. Econometric methods are taught in graduate classrooms without belabouring this obvious, implicit point ... However, when it comes time to solve real-world problems, ... every tool applies; whatever works, works" (Quah, 1995, p.1596). This conclusion, however, appears unconvincing. One cannot be satisfied by the fact that a method is simply *working* ; the question of theoretical interest is *how well* the method is working. This naturally leads into the domain of the interplay between theory and empirical evidence in macro-econometric models.

At the other extreme is the argument that economic theories should not be taken seriously either. "To some extent, the varying weights given to theory reflect the range of goals in applied work, from developing theory, to measurement with existing theory, to short-term forecasting. Given this assortment of goals, there is in turn a range of methods in use, with a continuum between econometricians and business cycle theorists" (Gregory and Smith, 1995, p.1607). It appears that this argument does not provide a satisfactory answer to the problem, yet it contributes to the understanding of the gap which exists between theory and observation by classifying the applied work in economics according to their goals.

Between these two extremes, lies the attempts to link theories and empirical methods in the practice of economic model building. Perhaps Hendry's approach is the most successful one since it is based on the dichotomy between mapping from theory into a model in economic analysis and mapping from a model into reality to discover the economic structures in econometric studies. "The best is that theory delivers a model which happens to capture structure after appropriate estimation. Since structure must be invariant under extension of the information set, it cannot be learnt from theory alone without an assumption of omniscience. However, structures potentially can be learnt from empirical evidence without prior knowledge as to what exists to be discovered. Hence, economic theory is neither necessary nor sufficient for determining structures, although it remains one of several useful tools for the econometricians" (Hendry, 1995, pp.1633-34). Hendry's argument does not pay proper attention to the theoretically vital question of what determines the dynamics of growth in economic theorisation in the interplay between theory and observation in applied econometrics.

In summary, we need to be wary of both theory-based astatistical calibrated models as well as data-instigated atheoretical models. The recognition of the gap between theory and observation has been conducive to the advancement of our knowledge about the problematic interplay of economic theorisation and applied econometric model building. This is likely to continue so. Having realised the possibility of a theory becoming a straitjacket, the econometricians, despite the teaching of the Cowles Commission's dichotomy of economic theory and econometric models, are now in a stronger position in practice to constantly revise and modify economic theories in order to make them more suitable for applied work. However, the lack of coherent theoretical justification is a real weakness which will persist in econometric practice for the foreseeable future.

2.6 Rational Expectations, the Lucas Critique and the Policy Ineffectiveness Debate

In an engineering approach to economic applications of optimal control theory, it is usually assumed that the behaviour of an economic agent does not depend upon anticipation of future events including the future course of policy actions. This is not the case, however, for recent advances in optimal control applications to economic systems where the private sector's expectations of government's future decisions play a significant role. Although expectations of future values of endogenous variables have long been recognised as an important topic in macro-econometrics, the traditional expectations-formation process has been usually confined to the conventional backward-looking dynamic or distributed lag models. The rational expectations hypothesis has shifted the expectations-formation process from an essentially backward-looking to a forward-looking perspective.

In order to explain how expectations are formed, Muth (1961, p.315) has advanced the hypothesis that they are essentially the same as the predictions of the relevant economic theory. He maintains that "expectations of firms (or, more generally, the subjective probability distribution of outcomes) tend to be distributed, for the same information set, about the prediction of the theory (or the objective probability distribution of outcomes)".[50] Muth's paper, which has motivated a rich literature on the subject, is largely based on his earlier paper (1960) in which he established an economic argument to rationalise the "adaptive expectations mechanism" advanced by Milton Friedman (1956) and Phillip Cagan (1956). The rational expectations hypothesis was then developed further and extended by the work of Lucas (1965, published in 1981; also 1966, published in

[50] Keuzenkamp (1991) refers to an early paper by Tinbergen (1932), published in German, which anticipates much of Muth's analysis on rational expectations. In this regard, the work of Grunberg and Modigliani (1954) should also be mentioned. They showed that economic agents can react to forecasts which might alter the course of events. They demonstrated how these reactions can be taken into account in order to produce "correct and self-fulfilling forecasts".

1981) and Lucas and Prescott (1971).[51]

Estimation methods available for macroeconomic models with rational expectations are usually based on the assumption that expectations variables coincide with the future solution values over a sequence of periods. In other words, treating expectations variables explicitly rather than substituting them by appropriate distributed lag functions, necessarily requires that expectations coincide with the conditional expectations of variables based on the model itself and on all the available information (model-consistent expectations).[52]

Recent advances in econometric models with rational expectations imply that this hypothesis mainly affects short-run properties of models. Wallis (1995) reports that "as methods of estimating rational expectations models have been incorporated into large-scale macro-econometric modelling practice, it has become clear that various important long-run model properties do not depend on the choice between forward and backward-looking dynamic equations; rather, this choice principally affects the model's short-run properties" (p. 342-343). Bikker, van Els and Hemerijck (1993) confirm this conclusion by estimating a Dutch model with rational expectations.[53]

The structure of the information set and the cost of acquiring more information are of prime theoretical concerns. It is usually assumed that individuals know the structure of the entire model as well as the historical values of all relevant vari-

[51]For comprehensive and critical reviews of rational expectations hypothesis see Shiller (1978), Fischer (1980), Begg (1982), Minford and Peel (1983) and Sheffrin (1983). For early contributions of the rational expectations hypothesis on macroeconomic policy (and particularly on monetary policies) see Sargent (1973, 1977), Sargent and Wallace (1973, 1975, 1976), Barro (1976), Fischer (1977) and McCallum (1977).

[52]For contributions to estimation of econometric models with rational expectations see Muth (1960, published in 1981), Lucas and Sargent (1979), Blanchard and Kahn (1980), Hansen and Sargent (1980), Chow (1980), Wallis (1980) and Pesaran (1987). Pagan (1986) provides a survey of the appropriate estimation methods for macro-econometric models with rational expectations.

[53]For problems associated with macroeconomic policy formulation using large econometric rational expectations models see, Christodoulakis, Gaines and Levine (1991).

ables. The rational expectations hypothesis implies that individuals do not make systematic forecast errors since the information set available to them includes the past errors. Since expectations are forecasts conditional upon the set of available information, the prediction errors are orthogonal to the information set. In a stochastic environment this means that the unobservable subjective expectations are exactly the mathematical conditional expectations which are derived from the model known to the individual.

The implications of rational expectations for policy analysis are remarkably significant. Unlike conventional backward-looking models, models with forward expectations differentiate between anticipated and unanticipated policy changes. In the case of anticipated shocks, some responses prior to the actual shock might be expected. The same argument applies for temporary and permanent shocks. In a backward-looking environment, the model tends back to its original state when the temporary shock is removed. However, a forward-looking expectations allows individuals to change their current behaviour on the basis of anticipating the future removal of a temporary shock.

Lucas (1972a,b) has received the credit for applying the rational expectations hypothesis to macroeconomic models. However, the full impact of the rational expectations hypothesis on economic policy analysis and optimisation did not take place until the work of Sargent (1973), Sargent and Wallace (1975), Barro (1976), Lucas (1976) and Kydland and Prescott (1977). Their work initiated the debate known as *policy ineffectiveness* in models embodying rational expectations. For example, Sargent and Wallace (1975, p. 242) demonstrated that in models with rational expectations "the probability distribution of output is independent of the deterministic money supply rule in effect". In other words, the anticipated systematic monetary policy could have no effect on the mean and variance of output. This is a very strong result. It generalises the claim of Friedman (1968) that in the long-run output was independent of monetary policy: Sargent and Wallace concluded that output was independent of monetary policy even in the short run.

A number of research work, based on the absence of sufficient future or contingent markets, were delivered to demonstrate that the anticipated systematic monetary policy does have some effects on output (see, for example, Buiter 1981). Also Stanley Fischer (1977) provides a model with rational expectations (based on sticky wages) in which systematic monetary policy can be used to stabilise the economy. However, Holly and Hughes Hallett (1989) have shown that the neutrality proposition of Sargent and Wallace is essentially associated with the concept of *controllability* in optimal control theory. In this context, the question of existence of optimal policy can be reduced to the controllability conditions in models with rational expectations.[54] The work of Sargent and Wallace (1975) was significantly influential not only for its profound contribution to the policy ineffectiveness debate but for illustrating the case where the solutions to macroeconometric models with rational expectations could be substantially different from the solutions obtained otherwise.

The assumption of optimising behaviour in expectations formation which is inherent in the rational expectations hypothesis ensures that the systematic forecast errors associated with alternative hypotheses (such as adaptive, regressive or extrapolative expectations which relate expectations on future values to past observations) are avoided. Although one might prefer a Bayesian predictor based on explicit optimising behaviour, the comparative convenience of empirical imple-

[54]Given a model of an economy and given the policy sequence at the disposal of the policymaker, the controllability is defined as whether it is possible to reach any desired policy objectives. Controllability is the necessary condition for the existence of optimal policies. See, for example, Kwakernaak and Sivan (1972).

mentation can be considered as the strength of the rational expectations hypothesis. However, the rational expectations hypothesis does not theoretically address important issues such as the followings: How does an economic agent construct the true structure of the economy used in forming the rational expectations? (particularly when there is not a general agreement amongst economists on how the economy functions in the real world). What are the learning and revision mechanisms by which economic agents optimise the process of expectations formation to avoid systematic forecast errors?

The work of Lucas (1976) fundamentally changed the process of policy evaluation. He demonstrated that the effects of different policy regimes on the reduced form coefficients of an econometric model, arising from private sector's expectations, were ignored in the conventional (backward-looking) approach. The economic agents' expectations play an important role in the Lucas critique. The structure of an econometric model depends on the optimal decisions of economic agents. Expected future behaviour of control variables effectively influence such optimal decision rules. Since expected values of policy variables vary with changes in policy regimes the structure of an econometric model will become dependent upon the policy rules. It should be noted that, as Buiter (1980, pp. 35-36) has observed, the Lucas critique does not necessarily rely on rational expectations; it is essentially based on the assumption that agents form expectations on their perceptions about the policy regimes.[55]

[55] "Private sector behaviour is influenced in many ways by expectations of future variables. If changes in government behaviour change these expectations, models that ignore such links from government behaviour via private expectations to private behaviour are likely to forecast poorly and to lead to misleading conclusions being drawn from policy simulations. This conclusion does not require Muth-rational expectations *per se*, only some direct effect of government behaviour on private expectations. The assumption of Muth-rational expectations provides the additional hypothesis that the link between private sector expectations and government behaviour comes through the private sector's knowledge of the true structure of the model, including the parameters that describe government behaviour" (Buiter 1980, pp. 35-36).

More specifically, the Lucas critique effectively refers to a significant weakness of econometric simulations to obtain predicted values of state variables and hence to provide a guidance for policy decisions. The essence of the argument is that the true parameters in an econometric model may vary with alternative policy sequences. For example, Lucas (1967) shows how a change in the variance of the money supply (a policy parameter) will change the slope of the Phillips curve (a structural parameter), implying that parameters of macro-econometric models that appear structural may not be invariant to changes in policy. Hence, Lucas (1976, p. 20) concludes that "simulations using these models, can, in principle, provide no useful information as to the actual consequences of alternative economic policies ... [This is] based not on deviations between estimated and *true* structure prior to a policy change but on the deviations between the prior *true* structure and the *true* structure prevailing afterwards".

Herein, lies the basic Lucas' problem for what he calls the "theory of economic policy". The classical methods of estimating "structurally stable relationships" suitable for simulation with alternative policy sequences presupposes the invariant character of estimated structural relations to policy rules: "To assume stability of [an econometric model] under alternative policy rules is thus to assume that agents' views about the behaviour of shocks to the system are invariant under changes in the true behaviour of these shocks. Without this extreme assumption, the kinds of policy simulations called for by the theory of economic policy are meaningless" (Lucas, 1976, p. 25). He adds that "Everything we know about dynamic economic theory indicates that this presumption is unjustified" (p. 25).

In evaluating the Lucas critique, Gordon (1976) discusses that Lucas' condemnation of economic policy evaluation is not only pessimistic but is rather considerably overstated. He concludes, (p. 47), that "the effects of *some* policy changes can be determined if parameter shifts are allowed and are either (a) estimated from the response of parameters to policy changes within the sample period, or (b) are deduced from *a priori* theoretical consideration". It is now agreed the Lucas critique has made it quite clear that policy evaluations cannot satisfactorily be performed within the *classical* theory of economic policies. In other words, existing econometric models cannot be used for examination of policy changes since the structural parameters, such as the propensity to consume out of wealth or the interest elasticity of money demand, would likely change as policy changes.[56]

Nonetheless, the Lucas critique has inspired new empirical research work in macroeconomics to identify the *deep structural parameters.* It is agreed (see, for example, Sargent 1982) that once the truly structural parameters in an econometric model (i.e. tastes and technology in utility and production functions) are identified, the response of consumers and producers to policy actions can be deduced. Instead of estimating the structural relations, the parameters of, for example, utility function are estimated in this new approach. Intertemporal optimisation lies at the heart of this approach and the parameters (of, say, utility function) are estimated from the first order conditions which are in fact the Euler equation.[57]

[56]For more recent critical comments on the Lucas critique, see Sims (1982, 1987) which specifies the very restrictive conditions under which the identification of *differences* between system behaviour under different policy rules with *changes* in system behaviour when the policy rule is changed at some point in future is valid. Tony Lawson (1995) offers one of the most recent contributions to the Lucas critique. He generalises the Lucas critique beyond the simple implications for policy simulations using already derived econometric models. He attempts to extend the critique to the "endeavours of constructing and estimating such models in the first place, if using observations recorded in periods in which policy rules have been frequently changing" (p. 258).

[57]In this regard, Hall and Mishkin (1982) have estimated the rational expectations life-cycle consumption model using panel data in order to arrive at the parameters' estimation of a utility function. This method can also be used to test restrictions. For example, by testing restrictions imposed by the underlying model of intertemporal optimisation, Hall and Mishkin (1982) have found that about 20 per cent of consumers in their sample do not satisfy the first order Euler conditions, implying that they may be regarded as being liquidity constrained. For a further

Despite the fact that the rational expectations hypothesis and particularly the contributions by Lucas, have had profound effect on both policy modelling and econometric practice, in general and the mechanism of building policy-oriented macro-econometric models, in particular, it is by no means acceptable that all areas of macro-econometric behaviour should be modelled according to forward-looking or rational expectations hypothesis. As Currie and Levine (1993, pp. 1-2) have pointed out "Even if we were convinced that all economic agents behaved in this way, backward-looking relationships can be regarded as empirically acceptable approximations if the influence of the past on current decisions greatly outweighs the influence of future (rational) expectations. The stability of many macroeconomic relationships in the face of many changes in regime indicates that, for whatever reason, the Lucas critique may not be all pervasive".

2.7 Time Inconsistency and the Optimal Control of Macro-econometric Models with Rational Expectations

The problem of ensuring consistency in dynamic choice was first addressed in a seminal paper by Strotz (1956). The problem for Strotz was not to explore the implications of rational expectations for optimal policy-making. He was mainly concerned with examining the problem of optimal choice among a number of alternative time-paths for consumption when consumer's taste changes. For Strotz the crucial question was this: "if [a consumer] is free to reconsider his plan at later date, will he abide by it or disobey it- *even though his original expectations of future desires and means of consumption are verified?*" (p. 165, emphasis in the original). His answer is that the optimal plan of future behaviour chosen as of a given time "may be inconsistent with the optimising future behaviour of

discussion of this point and a strong critical analysis of the Euler equation approach (mainly with reference to its identification problems), see Garber and King (1983).

the individual (*the intertemporal tussle*). In this case, (1) the conflict may not be recognised and the individual will then be *spend-thrifty* (or *miserly*), his behaviour being inconsistent with his plans, or (2) the conflict may be recognised and solved either by (a) a *strategy of pre-commitment*, or (b) a *strategy of consistent planning*" (p. 180, emphasis in the original). It is interesting to note that the concept of *pre-commitment*, which is now widely used in optimal control of macro-econometric models with rational expectations, was originated and carefully applied by Strotz in 1956.

Note that for the case of consumer behaviour considered by Strotz, a contractual savings scheme could enable an individual to enter into pre-commitment to carry out the plan. It should also be noted that by the strategy of consistent planning, Strotz implies that the optimal current decisions are based on the assumption that plans will be revised in the future. The dynamic programming approach can essentially be used to formulate a consistent planning by imposing the principle of optimality at each stage of planning.

The problem of inconsistency in dynamic choice was extended by Pollak (1968), Blackorby *et al* (1973), Pelag and Yaari (1973), and Hammond (1976), among others. However, it was the seminal work of Kydland and Prescott (1977) which first recognised the possibility of optimal policies in models with rational (forward-looking) expectations to become sub-optimal through the passage of time (see, also Prescott, 1977). This property, which is known as *time inconsistency* or *dynamic inconsistency* in forward-looking models, has significant implications for optimal control applications to economic policy-making practice.

More precisely, Kydland and Prescott (1977) have shown that when government economic policies affect the way in which the private sector's expectations are formed, an optimal future policy obtained in one period will not be optimal from the vantage point of that future period, implying that *ex ante* optimal policies become sub-optimal *ex post.* In other words, when the current state is a function of the announced future economic policies, time-consistent values for current and future policy-instruments are not optimal because a lower value for the objective (cost) function can be achieved by implementing a different value for the future policy-instruments. It is, therefore, to the advantage of the policy-maker to change the policies in subsequent periods which are in optimal from the vantage point of a previous period. This conclusion is based on the assumption that economic agents take into consideration any relevant information (including the information on the intentions of policy-makers) in the process of optimising their dynamic choice.

For *causal* or backward-looking models in which the current state is a function of both the past state variables and the current values of control variables, the application of the standard optimal control techniques, such as Bellman's dynamic programming, does not face any problem. However, since the standard dynamic programming does not accommodate the impact of future policies on current values of state variables, the principle of optimality breaks down for optimal control of *non-causal* or forward-looking models in which current state variables depend on the anticipated future states.

To explain the failure of the principle of optimality recall that, in engineering optimal control the current state of a system, in a non-stochastic environment, is a function of the initial state and the sequence of policies applied, while the future state of the system depends upon the current policy and the current state. The private sector will respond to its own expectations of the future state of the economy resulting from the policy actions announced by the government. The principle of optimality maintains that "an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision"(see Bellman, 1957). The assumption of forward-looking rational expectations hypothesis clearly violates the additive separability assumption of objective function inherent in the principle of optimality. When policy optimisation unfolds, previous optimal policy recommendations may not appear to be optimal due to the observed variations in state variables resulting from the realisation of forward-looking expectations. The dynamic programming solution, using backward recursive functional, does not provide the same answer as the global optimum values unless the model is *causal*. In other words, the time-consistent solution is not necessarily the same as the optimal solution unless the future decisions do not affect the current states, which could make the system causal.[58]

The above argument underlies the strong reservation of Kydland and Prescott (1977) on the application of optimal control theory to economic stabilisation policies. They argue that "even if there is an agreed-upon, fixed social objective function and policy-makers know the timing and magnitude of the effects of their actions, discretionary policies, namely, the selection of that decision which is best, given the current situation and a correct evaluation of the end-of-period position, does not result in the social objective function being maximised. The reason for this apparent paradox is that economic planning is not a game against nature, but, rather, a game against rational economic agents. We conclude that there is *no* way control theory can be made applicable to economic planning when expectations are rational" (p. 473).

If Kydland and Prescott's recommendation is not to attempt to select policies

[58]For further discussions on this point, see Holly and Zarrop (1983), Levine and Holly (1987) and Holly and Hallett (1989). For an examination of the conflict between optimality and time-inconsistency, see Calvo (1978) who carefully examines the difference between the constraints facing the policy-maker in subsequent periods and what they were in the previous period.

optimally, how should policies be selected? They join Lucas (1976) in saying that "economic theory [should] be used to evaluate alternative policy rules and that one with good operating characteristics be selected. It is probably preferable that selected rules be simple and easily understood, so it is obvious when a policy-maker deviates from the policy. There could be institutional arrangements which make it a difficult and time-consuming process to change the policy rules in all but emergency situations" (p. 487). In what follows we examine the pattern of research work which has developed in response to the new outlook towards optimal policy design originated by the seminal work of Kydland and Prescott.

The underlying point is that the optimal policy choice is formulated as a dynamic game between intelligent players, i.e. policy-makers and the private sector. Under such circumstances, the behaviour of the private sector is conditioned by its perception of the nature of the control to be applied. When the policy-maker announces the policies, and when the private sector's expectations are formed, there is always an incentive for the policy-maker to renege on the previously announced policies (time-inconsistent optimal policies); and since this can be anticipated by the private sector, the announced policies lack credibility in the absence of a pre-commitment mechanism. A simple solution follows that the government's discretionary power should be taken away by binding it to a fixed policy rule.

It is important to note that the failure of the standard optimal control theory in obtaining optimal policy sequence for dynamic economic systems with forward-looking expectations is not the result of the structural shortcoming of control theory in handling systems which actively react to control signals. One can argue that the prime interest of many econometricians and mathematical economists in the late 1960's and early 1970's were basically in the automatization of optimal economic planning, the computations of optimal policies in econometric models, and the examination of mathematical properties of optimal policy and state trajectories. This overshadowed the very basic argument of the structural differences between physical and economic systems and hence largely ignored those properties specific to economic systems which usually constrained the applications of engineering control theory. Otherwise, mathematical economists and econometricians did not have to wait until the 1980's and early 1990's to approach the dynamic optimisation of economic systems from a game theoretic point of view. Mathematical control theory had so much to offer in game-theoretic control theory as early as the late 1950's.[59]

In the context of a two-person (government and the private sector), non-cooperative closed-loop game in which policy rules at any time are functions of the state at that time and thus of the new information set which becomes available, each player's policies affect the state and, therefore, influence the future policies via the closed loop or feedback mechanism. Each player minimises his cost function by choosing his optimal policies subject to equations of motion and the rules of the game (for example, *Stackelberg* or *Nash* assumptions for single-stage or one-shot games).

With regard to the hierarchical structure of the Stackelberg games in which a leader (the government) can impose his actions on a follower (the private sector), the dominant player anticipates the reactions of the follower to announced policies and then optimises accordingly. The government optimises subject to the private sector's first-order conditions for optimality. However, Miller and Salmon (1983, 1984) have shown that, in contrast to the single-controller, the optimal rule cannot be expressed as a linear time-invariant feedback even if an open loop Stackelberg game is assumed. They have demonstrated that, in this case, optimal policies are

[59]See Berkowitz and Fleming (1957), Ho, Bryson and Baron (1965) and Behn and Ho (1968) for an approach based on differential games and optimal pursuit-evasion strategies.

either a linear *time-varying* contingent rule or a type of *integral control.*

In this regard, Levine and Currie (1984) argue that such rules are attractive because they avoid feedback on other possible unknown variables. The open loop Stackelberg equilibrium exists if the leader is committed to pursue his announced optimal policies. As discussed earlier, under the assumption of unrestricted discretion, the dominant player can benefit by reneging on the announced initial plan since the leader knows that his announced future plans can significantly influence the follower. However, after the expectations are formed, bygones are bygones and the departure from the *ex ante* optimal policies becomes more profitable . Note that the plan which involves the combination of announcement and reneging is dynamically consistent since there is no incentive to depart from it (the *perfect cheating* solution in Currie and Levine, 1985).

The formal procedure of incorporating rational expectations into the optimal control problem is to partition the state vector into predetermined state variables and a vector of non-predetermined forward-looking (free or jump) variables.[60] Levine and Currie (1983, 1984) and Currie and Levine (1983, 1984a,b,c) have obtained the complete solution to control rules in rational expectations models. They have found that a feedback rule on the predetermined state variables must be in the form of either a linear time-varying rule on the current value of the predetermined state variables or a time-invariant rule on the current and past values of such variables.

When the two players (government and the private sector) interact according to Nash assumption, each player takes the other's actions as given and thus each has equal status in the game. In other words, under the Nash assumption, each player has no influence on the behaviour of others and hence an equilibrium posi-

[60]For the exact definition of these terms, see Currie and Levine (1982) and Buiter (1984).

tion is one that provides no incentive for players to move. Whilst the Stackelberg assumption appears to be more satisfactory in explaining government-private sector's behaviour in macro-econometric models, the policy-maker, under the Nash assumptions, should ensure that the expectations are consistent with the optimised policies since with rational expectations hypothesis economic agents understand and correctly anticipate the policy-makers' policies. In an open loop Nash game, each player minimises his cost function subject to the system dynamics while treating as parametric his rivals' policy vector.

Currie and Levine (1985) provide a solution to the open loop Nash game in the context of optimal control theory. The Nash assumption ensures the identity of *ex ante* and *ex post* optimal policies since there will not be an incentive to renege on the announced polices. The Nash game precludes time-inconsistency by making players ignore the effects of their decisions on the rivals' policy rules.[61]

2.7.1 Time Inconsistency, Reputation and the Stochastic Environment

Repeated games involve memory and thus current strategies depend upon the past history of the game. The resulting notions of *reputation* and *credibility* may prevent the government to depart from the announced optimal plan. The problem of reputation building and its implications for time-inconsistency are reported in Barro and Gordon (1983). Backus and Driffill (1985a,b) and Barro (1986) provide a number of important extensions and Currie and Levine (1993) present the most recent advances on this issue.

The essential point is that designing a superior policy to the Nash strategy is possible only when the policy-maker is concerned with reputation. The contributions on reputational considerations can be regarded as an important development

[61]For a survey of Nash and Stackelberg equilibrium strategies in dynamic games, see Cruz (1975).

in the time-inconsistency literature. The mechanism of reputation building is an interesting complex problem. For example, Backus and Driffill (1985a,b) refer to the implausible length of time that the government and the private sector have been playing the game in the UK experience of anti-inflationary policies. The UK government's commitment to such policies has been met with private sector scepticism to expect higher inflation. The behaviour of the private sector has not reflected the commitment of government to anti-inflationary policies.

Much of what has been said earlier for the deterministic case can be applied to the more realistic stochastic environments. Levine and Currie (1984) have shown that the *certainty equivalence* property of linear systems with quadratic cost function applies equally to the optimal control problems with rational expectations. In other words, the optimal control rule is the same for the stochastic and deterministic cases when the system is linear and the cost function is quadratic. In a further contribution to optimal control of stochastic rational expectations models, Currie and Levine (1985) have pointed out the significance of the discount parameter on the time inconsistency property of the optimal policy in a stochastic environment. When the policy-maker gives higher weight to the future the incentive to renege on the announced policies declines. The policy-maker should weigh the advantages of reneging against the costs of pursuing the inferior time -consistent rule with respect to future shocks. As Currie and Levine (1985) conclude, if the rate of discount is not too high, the future cost will outweigh the gains from reneging. This implies that the policy-maker has an incentive to adhere to the ideal rule and thus avoid the temptation to cheat. Since the private sector is aware of this, the ideal rule is credible and sustainable.

It is well-known that stochastic shocks transform a deterministic policy game into a repeated game. The policy-maker is, therefore, more inclined to invest in its reputation to secure long-term policy gains rather than short-term gains from reneging. However, as Currie and Levine (1985) have pointed out "the key assumptions are that governments last for ever and that the private sector never forgets past inconsistencies so reputations cannot be re-established".

There are arguments suggesting that in a stochastic environment an observed policy change should be decomposed into two components: one arising from the optimal response to random shocks (stabilisation component) and the other, arising from the time-inconsistency of the optimal policies (strategic component). Although such an uncertainty about the cause of an observed policy change is beneficial to the policy-maker by exploiting the private sector's uncertainties, the private sector might become more inclined to distrust any government announcements due to the resulting higher levels of confusion. Canzoneri (1985) provides an example of this type of problems by examining monetary policy games and the role of private information.

A stochastic environment sheds more light on the question of simple rules versus full optimal feedback rules. Recall our earlier discussion on Kydland and Prescott (1977) who strongly advocated simple rules on the basis that they have good operational characteristics and can easily be understood, allowing private sector to monitor policy-makers' deviations from the announced policies. The second property is of crucial importance particularly when the private sector's behaviour depends strongly on their *understanding* of government's announced policies.

As Levine and Currie (1984) have shown, simple rules which are specified at the initial period to be linear time-invariant feedback rules on the state variables do not satisfy the property of certainty equivalence. One might, therefore, argue that the performance of these rules becomes a function of the nature of the future stochastic unknown disturbances. This can significantly reduce the desirability of simple rules in practical implementations. In a series of papers, Currie and Levine (1983, 1984a,b,c) and Levine and Currie (1983, 1984) have reported the possibility of identifying simple rules that perform well in the presence of different random disturbances. Since simple rules are basically designed to protect against random exogenous shocks, they are sensitive to the nature of such disturbances. The information concerning a group of shocks specific to an economy is, therefore, of vital importance in designing simple rules.

2.7.2 Rational Expectations and Econometric Modelling in Practice

Lessons from the repeated-game literature together with the advances in reputational equilibrium, strategic behaviour involving memory, stochastic environments in game theory and information structure of games have substantially enriched the literature on economic policy optimisation with forward-looking expectations. An important question is how these ideas are relevant to as well as being useful in the practice of economic policy-making in the real world

In contrast to major *theoretical* advances in macro-economics during the past twenty years, it appears that macroeconomists in business and government have largely continued to base their forecasting and policy analysis on conventional medium to large-scale macro-econometric models. To explain the disparity between the theoretical macro-economics and applied macro-econometrics one may argue that the nature and complexities of these theoretical developments have been of the sort that cannot be quickly adopted by applied macro-econometricians.

Mankiw (1988, p. 437-438) provides an analogy from the history of science to support the above argument: "The Copernican system held out the greatest promise for understanding the movements of the planets in the simplest and intellectually most satisfying way. Yet if you had been an applied astronomer, you would have continued to use the Ptolemaic system. It would have been foolhardy to navigate your ship by the more promising yet less accurate Copernican system. Given the state of knowledge immediately after Copernicus, a complete separation between academic and applied astronomers was reasonable and indeed optimal". It appears that Mankiw's argument is not very convincing since it is based on the assumption that the real value of the recent theoretical advances in macro-economics will ultimately be judged by whether they prove to be useful to applied econometricians. The validity of this argument can be seen only through the passage of time. In other words, what Mankiw is really suggesting is that we should wait to see its "success".

Computational difficulties associated with optimal control solutions for models with forward expectations have been one of the real obstacles in applied work. Recall that an optimisation algorithm usually requires a Jacobian matrix of first derivatives of targets with respect to control variables in all periods. In contrast to conventional backward-looking models, where target variables respond only to current and lagged values of control variables, in models with forward expectations, such derivatives require the evaluation of model-consistent forward expectations.[62]

Optimisation of large scale non-linear models with rational expectations can become excessively expensive from computational point of view. Obtaining linear representations of the original non-linear models is a reasonable attempt to deal with non-linear models. For example, Christodoulakis, Gaines and Levine (1991) have linearized the London Business School model before developing a methodology to design optimal fiscal and monetary policies for large econometric models

[62]For a further discussion of this point, see Wallis (1995).

with rational expectations.[63]

It is now generally agreed that despite its immediate low practical returns, the optimal control of macro-economic models with forward-looking expectations has considerable potential in advancing the econometric practice of model building. By emphasising micro-foundations as well as macro-economic policy coordinations (at national and international levels), macro-econometric modelling with rational expectations provides a better understanding of the real economy at work. It may also provide a profound impact on changing the outlook and the way economic policies are being designed. The two concepts of time-inconsistency and credibility are the most important outcomes when optimal control of an economic system is viewed as a dynamic game between intelligent players. These concepts will continue to play a significant role in an econometric approach towards optimal economic planning for the foreseeable future.

2.8 Summary and Concluding Remarks

In this chapter we have been concerned with some of the main underlying reasons for using mathematical and engineering control theory in economic policy optimisation and the limitations of their applications. More specifically, we have identified the conditions under which economic applications of optimal control theory have produced results of value. The prospects for future cooperation be-

[63]For earlier work on implementations of rational expectations in the London Business School model, see Budd *et al* (1984). See also Levine (1988), Levine and Currie (1987) and Christodoulakis and Levine (1988) for further elaboration on this point. Hall (1986) presents a Nash-type computation in a model with a single forward expectations variable. For a Stackelberg approach in deriving an inflation-unemployment trade-off by optimal control in a model with three forward expectations variables, see Wallis *et al* (1987, chapter 3) and for an appraisal of such different computational methods, see Fisher (1992, chapter 7). Both the Bank of England [Easton and Matthews (1984)] and the Treasury [Spencer (1984)] have reported experimenting with models incorporating rational expectations. Similarly, Darby (1984) reports on the National Institute's work on macro-economic models with rational expectations. And finally, for an optimal control software for rational expectations models, see Gaines, al-Nowaihi and Levine (1989).

tween control theorists and economists are also considered. We have presented in this chapter an analytical and historical framework to examine the past 45 years of research work in this field as well as the partnership between control engineers and economists in applying control theory to economic policy optimization. The main concluding remarks are stated below.

1. The early applications of classical mathematical optimization techniques in economic analysis, appeared in the late 1920's and early 1930's, did not profoundly encourage the application of control theory to economic analysis.

2. Following the great depression of the 1930's the study of economic regulations and stabilisation policies became fashionable. The Keynesian contribution made the government the main economic regulator and stabiliser on the ground that an inherent tendency to generate stability and full employment does not exist in a capitalist economy. On the other hand, the servo-mechanism (self-regulating systems or automatic stabilisations) was being extensively discussed in the classical control theory of the 1930's, 1940's and the early 1950's. This encouraged the interest of economists in engineering stability theory. However, the direct economic applications of stability and self-regulation in the servo-mechanism did not take place until the 1950's with the contributions of the engineers Goodwin (1951), Simon (1953), Tustin (1953) and the economist Phillips (1954).

3. Stability-type applications of control theory to economics soon came to an end because classical control theory itself entered the new realm of modern control theory in the late 1950's and early 1960's through developments of Pontryagin's maximum principle and Bellman's dynamic programming. In this new perspective, any controller, in a dynamic system design context, should posses some optimality character. However, the ideas of feedback and closed-loop control systems from classical control theory remained an important analytical tool

in economic policy optimization.

4. The encouraged interest in mathematical economic growth theories which were stimulated by the rapid economic growth in advanced industrial countries in the 1960's produced a vast literature on optimal growth theories based on Ramsey-type models of the 1920's enhanced by contributions of von Neumann and Kantorovich together with theoretical advances in utility and social welfare functions. In this framework, the first order conditions in Pontryagin's maximum principle proved to be the most effective method in analyzing the optimality conditions in growth paths. Modern optimal control theory has performed its most significant contribution in the theory of policy optimization in the 1960's.

5. The 1970's is the decade of econometrics and control theory. Many control theorists became interested in econometrics and many econometricians applied optimal control techniques in their medium and large-scale linear and non-linear models. These applications had become so fashionable that overshadowed the very interesting and original techniques, developed quite separately from control theorists, by economists such as Theil (1964), to study the properties of optimal economic policies.

Whilst the applications of deterministic optimal control to linear models with quadratic objective functions produced computationally tractable optimum state trajectory and policy sequence, applications of stochastic and adaptive control to non-linear econometrics models proved to be only an academic exercise in the literature of the late 1970's and early 1980's. Due to the accumulated forecasting errors and model misspecification which cannot be effectively treated by stochastic optimal control techniques, policy-makers were not in a position to prefer econometric control models to their classical simulation models on the ground that the latter could easily accommodate policy-makers' value judgements.

6. The applications of more advanced control techniques, such as dual control and estimation or probing and caution in stochastic control, only proved to be of theoretical interest and cannot constitute a significant improvement in the quality of optimal policies or system performances in economic analysis.

7. By the early 1980's, the basic techniques of optimal control theory, i.e. deterministic, stochastic and adaptive controls together with their computational and estimation applicabilities to econometric models of optimum policy design were carefully analysed by economists and their potentials and shortcomings had been fully identified. Research work since then has been directed either towards detailed examinations of economic policy issues within the well-known standard control techniques, or have been motivated by securing economic applications of optimal control theory in the presence of rational expectations hypothesis.[64]

8. Control theory, together with fundamental concepts in systems theory such as controllability and observability, remain valuable tools in analyzing dynamic properties of econometric models as well as in measuring the sensitivity of model specifications with respect to different patterns of priorities and estimation techniques.

9. The idea of automation has always been the underlying theoretical basis of engineering optimal control. This is in sharp contrast to the impossibility of automation in economic planning practices. When, following Keynes, the government was seen as the main controller, optimal control techniques were most warmly welcomed by the community of mathematical economists who viewed eco-

[64] In this regard, in addition to a number of journal papers referred earlier, the following books should be mentioned: Holly, Rustem and Zarrop (1979), Bensoussan, Kleindorfer and Tapiero (1980), Liu (1980), Kendrick (1981), Murata (1982), Feichtinger (1982), Feichtinger (1985), Aoki (1987), Carraro and Sartore (1987), Martos (1987), Holly and Hughes Hallett (1989), Beavis and Dobbs (1990), Gabszewicz, Richard and Wolsey (1990), Petit (1990), Kamien and Schwartz (1991), Leonard and Vanlong (1992) and Currie and Levine (1993).

nomic planning for stabilisation and growth simply as a mathematical theory of optimization.

The definite success of engineering control theory in achieving the high accuracy in guidance systems, in control of industrial processes, and in electronic systems did not provide any opportunity for the early Keynesian economists to think seriously about the underlying differences between physical and economic systems. Economic policy optimization are dynamic systems in which the feedbacks between government decisions, as the main controller and private sector's reactions to such controls are decisive. Economic systems are adaptive in nature in which actions and reactions constitute the mechanism of economic behaviour. By revealing such structural differences, the rational expectations hypothesis has directed control theory treatments of economic policy optimization to the new realm of game-theoretic approach. Control theory as a powerful *optimization technique* has enriched the theoretical analysis of optimal properties of economic policies since the 1960's and will continue to do so.

10. The increasing evidence that the traditional models did not adequately represent the data led to a growing theoretical emphasis on dynamic specifications and model selection in econometrics. This provides an example of how the data inconsistency could enhance further developments in econometric analysis. Moreover, theoretical developments in rational expectations, time-inconsistency and dynamic optimisation persuaded the theorists to believe that the traditional econometric models cannot successfully represent the theory. This led to the Lucas critique. The response of mathematical economists and econometricians were, respectively, to give much greater priority to represent theory at the expense of statistical analysis and to represent data at the expense of theory. The theory-based calibrated models and the data-based vector autoregressions are the two

well-known manifestations of this response.

11. The significant step towards the unification of theory and observation in applied work is the dynamic optimisation approach which provides a direct link between the neoclassical optimisation theory and applied econometric models. This approach, which is developed as a response to the rational expectations hypothesis and the Lucas critique, gives more emphasis to the role of economic theory in econometric modelling. All the behavioural equations, in this approach, are obtained directly as solutions to the dynamic optimisation problem facing an economic agent.

The crucial role of economic theory in the unification of theory and observation, advocated strongly by Smith and Pesaran, can basically be summarised as the replacement of the unobservable shadow prices by a functional form from observable variables. This allows economic theory to work outside the framework of dynamic optimisation not only to relax the computational aspects of estimation for prediction but to permit other relevant economic variables to enter optimisation problem through their influence on the shadow prices. While admitting that Smith and Pesaran's contribution is a significant advance towards strengthening the link between mathematical economics and econometrics, they have not justified the use of optimising models in building a bridge between theory and observation. Despite the fact that the dynamic optimisation models offer a substantial alternative within the framework of theory-based models, the assumption of representative agent, among the other serious restrictive assumptions, is a strong impediment in bridging the gap between theory and observation in applied econometrics.

13. The recognition of the gap between theory and observation has been conducive to the advancement of our knowledge about the problematic interplay of

economic theorisation and applied econometric model building. Having realised the possibility of a theory becoming a straitjacket, the econometricians, despite the teaching of the Cowles Commission's dichotomy of economic theory and econometric models, are now in a strong position in practice to constantly revise and modify economic theories in order to make them more suitable for applied work. However, the lack of coherent theoretical justification is a real weakness which will persist in econometric practice for the foreseeable future.

14. Attempts to accommodate rational expectations in control theory applications to economic policy optimisation necessarily involve game theoretic ideas. The amount of information that each player (government as the controller and the private sector) has on the constraints of the other player is an important factor in deriving the optimal solution. The fact that the public has considerable information on government's objectives and constraints in a democratic environment significantly limits the success of any set of optimisation policies. Moreover, despite being Pareto-optimal, the assumption of a cooperative game is not always realistic. On the other hand, there is no possibility of stopping a player from reneging on the cooperative solution or from cheating in a non-cooperative setting. Although the loss of reputation when the government reneges on the announced policies might ease the severity of this problem, the non-optimality of the policies adopted will remain a crucial issue.

15. When the government plays the dominant role in a single-stage non-cooperative game, the optimal policies will become time-inconsistent and thus violate the principle of optimality. The incentive to renege on the announced policies transforms the conventional optimal control of economic systems to the problem of finding policies which are optimal within the subset of credible and time-consistent policies. Lessons from the repeated game literature together with

the advances in reputational equilibrium, strategic behaviour involving memory, stochastic environments in game theory, and information structure of games have substantially enriched this literature.

16. Despite its immediate low practical returns, the optimal control of macro-econometric models with forward-looking expectations have considerable potentials in advancing the econometric practice of model building. By emphasising micro-foundations as well as macro-economic policy coordinations (at national and international levels), macro-econometric model-building with rational expectations provides a better understanding of the real economy at work. It may also provide a profound impact on changing the outlook and the way economic policies are being designed. The two concepts of time-inconsistency and credibility are the most important outcomes when optimal control of an economic system is viewed as a dynamic game between intelligent players. These concepts will continue to play a significant role in an econometric approach towards optimal economic planning.

Chapter Three

Optimal Consumption Behaviour Under Liquidity Constraints: an Application of Optimal Control Theory

3.1 Introduction

Recent empirical work using both micro and macro data have shown that consumption is excessively sensitive to current income than what is warranted by permanent income-life cycle hypothesis.[65] The excess sensitivity of consumption to income can be attributed, among other things, to the lack of a perfect capital market. The fundamental assumption of LC-PIH is that households maximise their lifetime utility functions subject to their lifetime budget constraints when free borrowing and lending to smooth consumption are possible. Imperfect capital markets are largely characterized by borrowing or liquidity constraints. A "liquidity constrained" household cannot borrow freely to smooth its consumption trajectory over time; thus current income becomes a major determinant of current consumption.

The importance and consequences of liquidity constraints in models of consumption behaviour are discussed in section 3.2. Optimal control of multi-stage

[65]See, for example, Hall (1978), Hall and Mishkin (1982), Flavin (1985), Zeldes (1989), Cushing (1992) and Jappelli and Pagano (1994).

dynamic model of consumption behaviour under liquidity constraints is the main focus of the remaining six sections in this chapter. A rational forward-looking consumer is assumed to behave according to a familiar Ramsey model with additively separable utility function. Having a span of life T, the consumer is assumed to have an initial financial wealth A_0 and receives a real disposable income Y_t in period t. He consumes C_t in period t and it is further assumed that his rate of time preference is δ. The real rate of interest r is assumed to be a deterministic function of time. Although the assumption of a known real rate of return is rather binding, except for the case where the government issues indexed bounds, it is used here for its mathematical tractability. The assumption of an additively separable utility function is also a restriction on consumer's preferences, but it is commonly used in the literature because of its analytical convenience. It is further assumed that there is no rationing in the goods market, so consumers are not constrained by the purchase of goods and services.

The problem facing the consumer is to find the optimal consumption path which maximises his expected lifetime utility function subject to a lifetime budget constraint and an additional constraint on borrowing. This leads to the following maximization problem in standard calculus of variation,

$$J = Max\, E \int_{t_0}^{T} e^{-\delta t} U[C(t)]dt, \tag{3.1}$$

subject to

$$A_{t+1} = (1+r)A_t + Y_t - C_t, \tag{3.2}$$

$$A_t \geq 0. \tag{3.3}$$

It is assumed that $U[C(t)]$ is increasing and strictly concave. Inequality (3.3) implies the existence of liquidity constraint. This means the consumer's end-of-period financial asset, after receiving income and allowing for consumption expenditure, cannot be negative. In other words, this inequality reflects the fact that consumers cannot consume today the income which they receive tomorrow. This condition can, of course, be generalised to

$$A_t \geq -a, \tag{3.4}$$

where a is the limit on net indebtedness. Further generalization will also be considered in this chapter.

To examine the role of liquidity constraints, it is useful to consider first the optimal consumption behaviour in the absence of liquidity constraints, i.e. where consumers can borrow and lend freely at the riskless rate. This will allow us to examine the effects of liquidity constraints on the optimal properties of the consumption path. In this case, the optimization problem reduces to the maximization of equation (3.1) subject to constraint (3.2). In section 3.4, optimal consumption trajectory and its properties are obtained by the method of *Pontryagin's maximum principle*. It is also shown, in this section, that such applications lead to the Hall's random walk hypothesis of consumption behaviour. Applications of *Bellman's dynamic programming* to optimal consumption path for the same problem is also examined in this section. We have shown that the application of the dynamic programming confirms Hall's hypothesis. Using the direct search technique based on Bellman's principle of optimality, we have obtained the basic recurrence functional equation for optimal consumption behaviour.

Generalization of the results obtained in sections 3.3 and 3.4 to the case where liquidity constraints are binding will then follow. Functional recurrence equation and envelope theorems are used in section 3.5 to derive the mathematical properties of the optimal consumption path when liquidity constraints bind. Section 3.6 deals with the application of the *generalised Hamiltonian* and the rejection of Hall's random walk hypothesis for liquidity constrained consumption functions. It is shown, among other results, that the application of control theory provides an explicit mathematical relation to examine the effects of liquidity constraints on consumption behaviour. Problems associated with interest rate variations and intertemporal elasticity of substitution in optimal consumption path under liquidity constraints are the subject of section 3.7. Section 3.8 considers the properties of optimal consumption in a stochastic environment, which is then followed by a summary and concluding remarks in section 3.9.

3.2 The Importance and Implications of Liquidity Constraints in Consumption Models

Liquidity constraints, as an explanatory variable in consumption models, necessarily introduce a number of interesting theoretical problems such as follows.

1. Liquidity constraints which represent the lack of *financial deepening*, can be regarded as an important determinant in consumption-saving behaviour in developing countries which are usually characterised by financial underdevelopment.

Defining financial deepening as increases in the ratio of financial assets to GDP and defining *excess sensitivity parameter* as the fraction of consumers who are more sensitive to current income than what PIH implies, one expects an inverse relationship between excess sensitivity parameter and financial deepening. This leads to the hypothesis that McKinnon's type of financial liberalisation [66] in developing countries, which would ease borrowing constraints, will reduce the excess sensitivity parameter. In this regard, one can argue that, in general, liquidity constraints make fiscal policies such as tax cuts or debt-financed fiscal spending more effective. A fall in current income affects consumption behaviour in developing countries more severely as compared to developed countries because a large portion of consumers in developing countries are liquidity constrained.[67]

[66] See McKinnon (1973).

[67] See, for example, Hubbard and Judd(1986) and Heller and Starr(1979).

2. The relationship between liquidity constraints and the aggregate saving rate is interestingly complex. The inability of households to borrow the desired amount in an imperfect credit market might lead to higher saving rates as compared with saving ratios in developed financial markets. This problem has received a considerable attention in recent literature on consumption-saving behaviour. Hayashi, Ito and Slemrod (1988) have examined this property for the United States and Japan; Jappelli and Pagano (1994) have attributed the high saving rates in Italy to its relatively underdeveloped consumer credit and mortgage markets; and Muellbauer and Murphy (1990) and Beyoumi (1991) argue that the sharp decline of the UK saving rates in the 1980s might be due to financial deregulations. By increasing saving rates, liquidity constraints might induce capital accumulation and hence can stimulate higher rates of growth.

The above conclusion might appear to be inconsistent with the McKinnon type argument that financial developments enhance the process of economic growth. McKinnon (1973) argues that by removing credit rationing, the resulting competitive financial intermediation promotes more efficient allocation of credit to investment and thus higher rates of return on capital can be achieved.[68] To reconcile the role of liquidity constraints in promoting growth rates with the McKinnon-Shaw model of financial liberalisation, one has to differentiate between credits to firms and credits to households [Jappelli and Pagano (1994)]. Such differentiation can be rationalised in view of the average loan size, informational asymmetries and the cost of contract enforcement. However, the argument that "if banks ration credits to households while making it available to firms efficiently, capital accumulation and growth will be enhanced"[69] might be valid only within a static framework. The maximising behaviour of firms will become adversely affected

[68]See McKinnon(1973). See, also Fry (1984), Bencivenga and Smith (1991) and Greenwood and Jovanovic (1990) among others.

[69]Jappalli and Pagano (1994), p. 84.

by the behaviour of liquidity constrained households in a dynamic optimization context through the resulting changes in households' demand.

3. Demographic structure of the population is likely to be a key factor in explaining the relationship between liquidity constraints and saving rates. Faster growth rates might stimulate the consumption of the young and thus reduce saving rates.

Liquidity constraints usually apply more severely to the young portion of population. The young usually find liquidity constraints more binding in smoothing the consumption pattern over time. This is a point of particular importance in modelling the nature and effects of liquidity constraints in developing countries due to their higher proportions of young population.

The existence of young population and the pervasive liquidity constraints in developing countries make the Keynesian type consumption function more data admissible. This does not, however, reduce the importance of life-cycle pattern in consumption-saving behaviour in these countries because savings during the middle years would still be optimal if consumers wish to enjoy the period of retirement. This explains the fact that fiscal policies in developing countries which affect the current income is more effective to influence the consumption-saving trajectory. Moreover, as Hubbard and Judd (1986) argue, it is the young segment of the population which strongly feel the implications of such current income fluctuations.

4. Liquidity constraints may become closely linked with the concept of informational asymmetries and the related notions of adverse selection and moral hazards. The reason for the banks' unwillingness to lend freely to households for consumption purposes might be the uncertainties about households' future

income as well as the risk of default. That explains why borrowing against the purchase of durable goods is not so binding because durable goods can be used as collateral.

Modelling the uncertainties associated with household's future income is currently one of the most active research work in theoretical developments in consumption theory. By adding the assumption of rational expectations to the standard life-cycle-permanent income hypothesis, Hall (1978) made the first significant attempt in formulating the stochastic implications of income in a consumption function. The household's decision on how much to borrow and save is not usually independent of the uncertainties about future events. This makes the question of insurance a matter of crucial importance in explaining saving-consumption behaviour in developing countries. The absence of an efficient system of insurance might further promote the precautionary saving motives. It follows that, as Besley (1993) maintains, savings, credits and insurance in developing countries are closely related with one another and can best be analysed within a unified theoretical framework.

Development of small scale indigenous financial institutions operating in rural and urban areas can partially relax household's liquidity constraints arising from their future income uncertainties. Such financial institutions may successfully administer the optimum allocation of loanable funds. The accuracy of information about potential borrowers will minimise the risk of adverse selection and moral hazards associated with credit allocation.

5. Household's uncertainties about their future income, a typical characteristic of developing countries, stimulate precautionary motives to save. These motives interact with liquidity constraints because in an underdeveloped credit market where households are usually unable to borrow when times are bad there exists

an incentive for higher savings in good times.

6. Although any relaxations of liquidity constraints through improvements in consumer loan markets permit an individual to increase his consumption, this incremental consumption should be paid back with interest during the consumer's life-cycle. Provided that the individual's real income does not grow enough, this interest payment will constrain individual's future consumption. Despite the fact that such a decline in future consumption might be consistent with individual's utility maximisations a tendency may exist for aggregate consumption to fall when the economy is not growing fast enough to compensate for the aggregate interest payments.

7. Assuming that liquidity constrained consumers generally are more sensitive to current income variations as compared to liquidity unconstrained individuals; and assuming further that liquidity constrained and unconstrained individuals are usually the monetary debtors and creditors, respectively, it follows that a significant increase in the real rate of interest might have income redistributional effect between borrowers and lenders. This will affect the level of aggregate consumption through the existing differences between marginal propensities to consume. Note, however, that in the absence of liquidity constraints, the effect of changes in the interest rate on consumption is usually expected to be minimal because the resulting intertemporal substitution and wealth effects work in opposite directions.

8. To the extent that the role of inflation on consumption behaviour is reduced to the effects of inflation-induced changes in interest rates on consumption, an increase in the rate of inflation may affect the liquidity constrained consumption through interest rate variations.

9. Provided that the nominal rate of interest and the nominal credit ceilings are

fully adjusted to accommodate the inflation rate, then consumption will remain unaffected because consumer's real wealth has not changed. If the credit limit for liquidity constrained consumers are not revised, they are forced in a position to reduce their consumption in proportion to any higher loan repayments resulting from the increased interest rates. To the extent that liquidity constrained individuals reduce their consumption, the aggregate consumption may decline following an inflation.

10. Under the circumstances that an inflation rate does not change the nominal rate of interest, the resulting fall in the real interest rate implies a redistribution of income from liquidity unconstrained lenders to liquidity constrained borrowers. The net effect on consumption appears to be indeterminate not because substitution and income effects work in opposite directions, but because there exists uncertainties on the future rate of inflation which hinders liquidity constrained consumers to increase their consumption in the first instance. However, as inflation proceeds, liquidity constrained consumers can increase their consumption.

11. The composition of consumer's asset portfolio is also important because the higher the degree of asset's liquidity in consumer's portfolio the lower would be the liquidity constraints. The purchase of illiquid physical assets (houses and lands) may affect the liquidity constrained consumption behaviour in the following ways: *i*) it reduces the portion of liquid assets in consumer's portfolio and hence increases the liquidity constraints, *ii*) it constitutes a collateral for borrowing and hence decreases the future liquidity constraints. Within this context, the problem of credit rationing appears to be of prime importance.

3.3 Optimal Consumption Properties using the Maximum Principle

The problem is to maximise the objective function (3.1) subject to the constraint (3.2). We employ the method of maximum principle due to Pontryagin (1962). A_t and C_t are state and control variables, respectively. In section 3.3.1, we first show the application of the maximum principle to consumer's intertemporal optimization problem produces an optimal consumption path which has the property that for an increasing consumption the rate of interest should exceed the consumer's rate of time preference. A Bernoulli-type utility function is used in section 3.3.2 to test the validity of the general result obtained in section 3.3.1. This example confirms the property $r > \delta$ along the optimal trajectory of consumption behaviour where r and δ are the real rate of interest and the rate of time preference, respectively. In section 3.3.3, we demonstrate how the Hall's important result obtained in 1978, i.e. the random walk hypothesis of consumption behaviour, can directly be obtained by applying the method of maximum principle to the classical maximization problem of consumer's utility function.

For applying the method of maximum principle in its continuous version, we first write equation (3.2), which in control terminology is usually called the *equation of motion* or *system dynamics*, as follows,

$$A_{t+1} - A_t = rA_t + Y_t - C_t.$$

The limit of the above equation when we take discrete periods of length Δt and then let Δt tend to zero, provides a continuous version of the equation of motion. Note that since our flow variables, rA_t, Y_t and C_t, are now rates per unit of time, the right hand side of the above equation should be multiplied by Δt. We, therefore, have

$$A(t + \Delta t) - A(t) = [rA(t) + Y(t) - C(t)]\Delta t.$$

Dividing by Δt and letting Δt go to zero, will give the time derivative of $A(t)$, i.e.

$$\dot{A}(t) = rA(t) + Y(t) - C(t). \tag{3.5}$$

Conventionally, we write t as a subscript in discrete models and as an argument in continuous cases. It is further assumed that the initial level of consumer's financial asset is given, i.e.

$$A(t_0) = A(0) = \alpha.$$

3.3.1 Consumer's Rate of Time Preference, Interest Rates and the Optimal Path of Consumption

Defining the multiplier $\lambda(t)$, which in control terminology is also called co-state variable, auxiliary variable, adjoint variable, or dual variable, we can construct the *Hamiltonian* as follows,

$$H = E\, U[C(t)]e^{-\delta t} + \lambda(t)[rA(t) + Y(t) - C(t)]. \tag{3.6}$$

Our control variable $C(t)$ can maximise H if

$$\frac{\partial H}{\partial C(t)} = 0,$$

or

$$E\, U'[C(t)]e^{-\delta t} = \lambda(t). \tag{3.7}$$

We observe that the maximization of the Hamiltonian gives the optimal consumption variable $C(t)$ not as a function of time but a function of the co-state variable $\lambda(t)$. The time path of the co-state variable is thus required. This entails solving a two boundary value problem in the *canonical equations.* From the canonical equations we have

$$-\frac{\partial H}{\partial A(t)} = \dot{\lambda}(t),$$

and by using equation (3.6) we have

$$\dot{\lambda}(t) = -r\lambda(t). \tag{3.8}$$

If we now solve equation (3.7) for $C(t)$ we obtain $C(t)$ as a function of $\lambda(t)$, which by substituting back into the equation of motion, i.e. equation (3.5), we derive an equation in terms of state variable $A(t)$ and the multiplier $\lambda(t)$. This together with equation (3.8) constitute a pair of differential equations for the state variable $A(t)$ and multiplier $\lambda(t)$. Note that due to the simplicity of our problem, $\lambda(t)$ can directly be obtained from equation (3.8), which by substitution into equation (3.7) and into the equation of motion (which can be written in terms of the state and the multiplier), provide the optimal trajectory for consumption and the resulting financial asset.

We are, of course, interested more on the properties of optimal control and state trajectories. To examine the optimal consumption trajectory, we differentiate equation (3.7) with respect to time. Since marginal utility of consumption $U'(C)$ is a function of consumption, and C is assumed to be a function of time, we have

$$E\,\frac{\partial U'[C(t)]}{\partial C(t)}\frac{E\,dC(t)}{dt}e^{-\delta t} - \delta e^{-\delta t}E\,U'[C(t)] = \frac{\partial \lambda(t)}{\partial t},$$

or

$$\{E\,U''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)]\}e^{-\delta t} = \dot{\lambda}(t).$$

Using equation (3.8) yields

$$\{E\,U''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)]\}e^{-\delta t} = -r\lambda(t).$$

We can now substitute equation (3.7) in the right hand side of the above equation to obtain

$$\{E\,U''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)]\}e^{-\delta t} = -E\,U'[C(t)]e^{-\delta t}r,$$

or

$$E\,U''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)] = -E\,U'[C(t)]r.$$

Dividing both sides by $E\,U'[C(t)]$ yields

$$\frac{E\,U''[C(t)]}{E\,U'[C(t)]} E\,\dot{C}(t) - \delta + r = 0. \tag{3.9}$$

Defining $\eta(C)$ as the elasticity with which the marginal utility of consumption declines as consumption increases, i.e.

$$\eta(C) = -\frac{C(t)U''[C(t)]}{U'[C(t)}, \qquad or \qquad \frac{U''[C(t)}{U'[C(t)]} = -\frac{\eta(C)}{C(t)},$$

we will have

$$-\frac{E\,\dot{C}(t) E\,\eta[C(t)]}{C(t)} = \delta - r,$$

or

$$\frac{E\,\dot{C}(t)}{C(t)} = \frac{r-\delta}{E\,\eta[C(t)]}, \quad or \quad E\,\dot{C}(t) = \left[\frac{r-\delta}{E\,\eta[C(t)]}\right] C(t). \tag{3.10}$$

The assumption of $CARA$ can be used to simplify equation (3.10). Define $\rho_a = -\frac{U''[C(t)]}{U'[C(t)]}$, or

$$E\,\rho_a = -\frac{E\,U''[C(t)]}{E\,U'[C(t)]} = \rho_a^*,$$

we can write equation (3.10) as follows,

$$\rho_a^* E\,\dot{C}(t) = r - \delta. \tag{3.10a}$$

Equation (3.10) together with the equation of motion, equation (3.5), provide a pair of differential equations in the state variable $A(t)$ and the control variable $C(t)$. Given initial values for consumption and financial assets, these two differential equations determine the subsequent changes along the solution trajectory which passes through this initial point. Since U is an increasing function and is strictly concave, therefore, $\eta(C) > 0$. Equation (3.10) implies that the optimal level of consumption increases, i.e. $\dot{C}(t) > 0$, if $r > \delta$, which means that the rate of interest exceeds the consumer's rate of time preferences. Note that the alternative case, i.e. $\delta > r$, implies that the consumer can improve his position by borrowing at the rate r. The optimum value of C will then decline at later stages in the life-cycle.

According to $r > \delta$, consumers will accumulate assets indefinitely, and in the limit, the income stream becomes irrelevant as consumption comes to be financed increasingly out of capital income. Liquidity constraints are unlikely to be of relevance for such consumers. In fact, *saving* and not *borrowing* is their concern. This does not, however, provide a serious problem since we can assume that in the presence of no inheritance or bequest, the discounted present values of consumption and income over the consumer life span must be equal. The optimality condition obtained above implies that the consumer should save during his early stages of life span and then spend income together with the accumulated assets during his later years. Again, the finding of Carroll and Summers (1989) should be taken into account: although one expects that in occupations with an uncertain but relatively flat income profile consumers should accumulate when they are young, but, as in most occupations, consumption tracks income closely.

3.3.2 The Bernoulli Case

Let us now consider the Bernoulli case, $U[C(t)] = lnC(t)$. This example demonstrates the validity of the result obtained in the previous section, i.e. along the optimal increasing consumption path the property $r > \delta$ holds. Despite the strong criticisms put forward by Arrow (1965, 1970) on the application of this type of utility functions on the ground that they are not bounded, the support of Samuelson (1969) among others, have made the use of Bernoulli type utility functions in economic analysis quite popular.

Consider a consumer with income $Y(t)$ who consumes $C(t)$ and whose financial assets is defined as in equation (3.5), which is repeated here.

$$\dot{A}(t) = rA(t) + Y(t) - C(t).$$

$A(t)$ and $C(t)$ are state and control variables, respectively. All other notations are defined earlier. The consumer wishes to maximise the present value of his additively separable utility function defined as follows,

$$J = Max \int_{t_0}^{T} e^{-\delta t}\, ln[C(t)]\, d(t). \tag{3.11}$$

As before, we first define the Hamiltonian,

$$H = ln[C(t)]e^{-\delta t} + \lambda(t)[rA(t) + Y(t) - C(t)]. \tag{3.12}$$

Differentiating H with respect to the control variable $C(t)$, yields

$$\frac{1}{C(t)} e^{-\delta t} - \lambda(t) = 0, \tag{3.13}$$

or

$$C(t) = \frac{1}{\lambda(t)} e^{-\delta t}. \tag{3.14}$$

Equation (3.14) gives the optimal consumption trajectory as a function of the multiplier $\lambda(t)$.

From the canonical equations we have

$$\frac{\partial H}{\partial \lambda(t)} = \dot{A}(t),$$

or

$$rA(t) + Y(t) - C(t) = \dot{A}(t),$$

which upon the substitution of equation (3.14) becomes

$$rA(t) + Y(t) - \lambda^{-1}(t)e^{-\delta t} = \dot{A}(t). \tag{3.15}$$

We also have

$$-\frac{\partial H}{\partial A(t)} = \dot{\lambda}(t),$$

or

$$-r\lambda(t) = \dot{\lambda}(t). \tag{3.16}$$

Equations (3.15) and (3.16) are a pair of differential equations which can be solved for financial asset $\dot{A}(t)$ and the multiplier $\lambda(t)$. First, we write the general solution of equation (3.16) as follows,

$$\lambda(t) = \lambda(0)e^{-rt}. \tag{3.17}$$

To find $\lambda(0)$, we substitute equation (3.17) into equation (3.15) to obtain

$$\dot{A}(t) = rA(t) + Y(t) - \lambda^{-1}(0)e^{(r-\delta)t},$$

or

$$\dot{A}(t) - rA(t) = Y(t) - \lambda^{-1}(0)e^{(r-\delta)t}.$$

Multiplying both sides by e^{-rt} yields

$$[\dot{A}(t) - rA(t)]e^{-rt} = Y(t)e^{-rt} - \lambda^{-1}(0)e^{-\delta t}. \tag{3.18}$$

Consider the term $A(t)e^{-rt}$. Taking differential with respect to t provides

$$\frac{d}{dt}[A(t)e^{-rt}] = \dot{A}(t)e^{-rt} - re^{-rt}A(t) = [\dot{A}(t) - rA(t)]e^{-rt}.$$

Substituting equation (3.18) into the above equation yields

$$\frac{d}{dt}[A(t)e^{-rt}] = Y(t)e^{-rt} - \lambda^{-1}(0)e^{-\delta t},$$

which by integration we have

$$A(t)e^{-rt} - A(0) = Y(t)[1 - e^{-rt}]/r - \lambda^{-1}(0)[1 - e^{-\delta t}]/\delta.$$

Since $A(T)$ is known, we can easily obtain $\lambda(0)$ from the above equation. This will complete our solution for optimal values of financial assets and consumption [from equations (3.15) and (3.16)].

Substitution of equation (3.17) into equation (3.14) yields the optimal trajectory for consumption, i.e.

$$C(t) = [\lambda^{-1}(0)e^{rt}]e^{-\delta t},$$

or

$$C(t) = \lambda^{-1}(0)e^{(r-\delta)t}. \tag{3.19}$$

Equation (3.19) implies that the optimal level of consumption grows if $r > \delta$. This is exactly the same result which we obtained in the previous section [see equation (3.10)]. The discussion on $r < \delta$ is similar to what we have said before and will not be repeated here.

3.3.3 Pontryagin's Maximum Principle and Hall's Random Walk Hypothesis

In this section, we show that the application of Pontryagin's maximum principle to consumer's optimal consumption behaviour will lead to Hall's random walk hypothesis of consumption behaviour. Similar results will be obtained in section 3.4 using Bellman's dynamic programming. Recall that equation (3.9) played an important role in deriving equation (3.10). Defining ρ_a as the Arrow-Pratt coefficient of absolute risk aversion $CARA$ [Arrow(1970), Pratt(1964)], we have

$$\rho_a = -\frac{U''[C(t)]}{U'[C(t)]}.$$

Substitute ρ_a in equation (3.9) to obtain

$$E\rho_a E\left[\dot{C}(t)\right] = r - \delta.$$

To find a discrete analogue to this equation, we can apply the mean value theorem. According to this theorem we have

$$E\left[\frac{C(t+h) - C(t)}{h}\right] = \dot{C}(t).$$

Equivalently,

$$E[C(t+h) - C(t)] = h\frac{(r-\delta)}{E\,\rho_a}.$$

Assuming $h = 1$ and $E\,\rho_a = \rho^*$ provide

$$C_{t+1} = \frac{1}{\rho_a^*}(r-\delta) + C_t + \xi_{t+1},$$

where it is assumed that $E(\xi_{t+1}) = 0$. Following Hall(1978), we assume that the interest rate equals the subjective utility discount rate, i.e. $r = \delta$, Hence

$$C_{t+1} = C_t + \xi_{t+1},$$

which is the random walk hypothesis of consumption behaviour.

The above argument proves our claim that the result obtained by the Hall-Euler equation approach in modelling optimal consumption behaviour, which has established an active field of research in consumption theory since Hall (1978), can be derived directly by applying Pontryagin's maximum principle to classical problems of maximising consumer's expected utility.

3.4 Optimal Consumption Properties using the Dynamic Programming

The analysis of discrete time dynamic multi-stage optimal consumption behaviour can best be analysed by using Bellman's dynamic programming. Consider a consumer with the following objective function

$$Max\, J = E\left[\sum_{t=0}^{T}(1+\delta)^{-t}U(C_t)|\Omega_t\right], \tag{3.20}$$

where $E[...|\Omega]$ is the expectation conditional on information set Ω at time t. δ is the individual's rate of time preference. The optimization process, therefore, starts by maximising the present discounted value of expected utility conditional on the information set available at time t_0.

The consumer's budget constraint is given by

$$A_{t+1} = (1+r)A_t + Y_t - C_t. \tag{3.21}$$

A_t is the consumer's financial assets measured at the beginning of the period.

The optimal consumption problem as proposed above, falls within the domain of dynamic optimization. The individual consumer can act at time t_0 to select C_0. He knows his initial assets at time t_0 to select C_0. Although the consumer knows his initial income Y_0, we assume there exists uncertainties about his income next period. He can only guess what that income will be and upon this he takes his decision on consumption. Hence, the optimal policy at this stage will not be independent of what the consumption will turn out to be in next period; the latter in turn will be a function of the optimal policy rule in the following stage. Such series of inter-related optimal policy formulations will continue as the process unfolds towards the end of planning horizon.

The dynamic programming formulation of optimal consumption behaviour is based on Bellman's principle of optimality which states that *an optimal policy has the property that whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision.*[70] The application of the principle of optimality reduces a multi-stage dynamic optimization into a series of two-period optimal decision rules. This does not, however, guarantee an explicit solution for the general optimal consumption behaviour. Such a failure would not imply that the solution does not exist. Stokey, Lucas and Prescott (1988) have proved the existence of the solution under quite general conditions.

There exists explicit solution for a special class of consumption problems whose utility functions belong to the $HARA$ class, i.e. hyperbolic absolute risk aversion. Recall that this class of utility functions include the isoelastic or constant relative risk aversion, $CRRA$ and the quadratic utility functions. Merton (1971) has proved that for the class of $HARA$ utility functions, the value function belongs to the same family of underlying utility functions. The explicit solution can be

[70]See R. Bellman (1957).

obtained once the parameters of the value function are derived.

The numerical solution for a non-linear dynamic programming formulation of an economic system is basically obtainable by using the return function and a backward induction. Consider a consumer at period $T-2$ where there are only two stages to go to the end of planning horizon. Given the value of financial asset A_{T-2}, the individual faces a two-period optimization problem in which the optimal decision on consumption should be derived. By solving this problem, the consumer obtains the optimal consumption as a function of wealth. From this, we can derive the expectation of utility from period $T-2$ onward. The value function for period $T-2$ can thus be obtained. Using this value function, the consumer starts solving the two-period optimization problem formulated in period $T-3$ which similarly provides the value function in period $T-3$. This process can be repeated backward until the initial period, where it is assumed that the initial level of consumer's financial asset A_0 is fully specified. By deriving the value function for period zero, we can obtain the associated policy value which is the optimal consumption policy at time zero. This can be used to determine the optimal level of financial asset in the next period. Such forward solutions continue until the optimal consumption trajectory for all periods are determined.

The organization of the remaining topics on optimal consumption properties using the method of dynamic programming is as follows. The application of direct search technique in deriving optimal consumption path is presented in section 3.4.1. This method is based on the repeated application of Bellman's principle of optimality . Despite the significance and importance of this technique in control theory, no attempt has been reported on the utilisation of this technique in optimal policy formulation of consumer's intertemporal choice problems.

The familiar Euler equation for optimal consumption is usually obtained by direct applications of the Lagrange method. In section 3.4.2 I have shown that the Euler equation can be obtained by using the functional recurrence equation together with the *envelope relation*. This approach provides more generalization by allowing the possibilities of dealing with problems such as stochastic income generating processes or stochastic interest rates.

The assumption of a Bernoulli-type utility function is made in section 3.4.3. By the application of the *guess technique* in solving the functional recurrence equation we have obtained the result that optimal consumption is a non-linear function of consumer's time preference and a linear function of wealth. Moreover, we have demonstrated how the guess technique can establish explicit relations for the value function. These results may generalise the result of Blanchard and Fischer (1990).

3.4.1 Optimal Consumption Path by Direct Search

We now examine the derivation of optimal consumption trajectory by direct application of Bellman's principle of optimality. For the sake of generality, a *non-linear model* for optimal consumption behaviour is assumed in which not only the consumer's utility function is non-linear, the equation of motion is also assumed to have a general non-linear structure, i.e. $A_{t+1} = f(A_t, C_t, t)$.

Let us mention at the outset that the existence of a numerical solution does not necessarily imply the possibility of obtaining such solutions. The central role of the application of the principle of optimality in optimization technique is that instead of searching among the set of all admissible policies we consider only those which satisfy the principle of optimality. This considerably reduces the number calculations. The number of calculations required by the direct enumeration increases exponentially with the number of periods in the optimization process, whereas the computational requirements associated with the principle of optimality increases linearly. Unfortunately, the memory requirements in such numerical solutions increases so fast that, except for problems of very low dimensions, increasingly higher computer capacities are required. This is usually referred to as the *curse of dimensionality*, a term invented by Bellman (1957).

Let us refer to two points before we examine the underlying mathematics of backward and forward solutions to value functions. The first point is that the value functions, which are alternatively called return functions, cost functions, or cost-to-go functions, are *optimal value functions* as their definitions imply. The second point is that since the recurrence property, as defined below, does not depend on stochastic properties of an optimization problem, we can, for notational simplicities, consider the deterministic case only.

For a greater generality, assume that the asset at the end of planning period, A_T, which is the state variable, is given. Define V_0 as the *cost* of reaching the final value for state variable A_T, thus

$$V_0(A_T) = \psi(A_T). \tag{3.22}$$

The problem is to find an optimal closed-loop policy sequence for consumption in the following general form

$$C^{opt}(A_t), \qquad t = 0, 1, 2, \cdots, T-1,$$

which minimises the following value function

$$V_N(C_t, A_0) = \sum_{t=0}^{T-1} \phi(A_{t+1}, C_t, t) + \psi(A_T), \tag{3.23}$$

where $C^{opt}(\cdots)$ stands for optimal consumption and V_N indicates that the consumer operates the optimization process over the N periods still to go. The form of the value function ϕ indicates that the consumption decisions to be made and included in V are C_0, C_1, C_2, $\cdots$, C_{T-1} and the resulting assets are A_1, A_2, $\cdots$, A_T.

Our approach to obtain the optimal consumption sequence is to solve a recurrence functional equation which is based on Bellman's principle of optimality.

Assume that the consumer is in period $T-1$ where there is only one period to go. The consumer should decide on his final period consumption policy based on the existing assets, A_{T-1}. The value function is

$$V_1(A_{T-1}, C_{T-1}) = \phi(A_{T-1}, C_{T-1}) + \psi(C_T), \tag{3.24}$$

which by using equation (3.22) becomes

$$V_1(A_{T-1}, C_{T-1}) = \phi(A_{T-1}, C_{T-1}) + V_0(A_T), \tag{3.25}$$

where V_1 indicates that there is only one period to go. Using the equation of motion, we know that A_T is a function of A_{T-1} and C_{T-1}, thus

$$V_1(A_{T-1}, C_{T-1}) = \phi(A_{T-1}, C_{T-1}) + V_0[f(A_{T-1}, C_{T-1})]. \tag{3.26}$$

The optimal consumption $C^{opt}(A_{T-1}, T-1)$ is that policy which generates the optimal value function, thus the optimal value function can be defined as

$$V_1^{opt}(A_{T-1}) = \max_{C_{T-1}} \{\phi(\cdots) + V_0(\cdots)\}.$$

The term $C^{opt}(A_{T-1}, T-1)$ means that the optimal consumption has been applied at stage $T-1$ with A_{T-1} as the consumer's asset.

The *cost* of operation or the return function over a two-period process with initial asset A_{T-1} is the following,

$$V_2(A_{T-2}, C_{T-1}, C_{T-2}) = \phi(A_{T-2}, C_{T-2}) + \phi(A_{T-1}, C_{T-1}) + \psi(A_T),$$

which upon using equation (3.24) yields

$$V_2(\cdots) = \phi(A_{T-2}, C_{T-2}) + V_1(A_{T-1}, C_{T-1}). \tag{3.27}$$

The optimal consumptions C_{T-1}^{opt} and C_{T-2}^{opt} are those policies which generate the optimal return over the interval $[T-(T-2)]$ defined as follows,

$$V_2^{opt}(A_{T-2}) = \max_{C_{T-1},C_{T-2}} \{\phi(A_{T-2}, C_{T-2}) + V_1(A_{T-1}, C_{T-1})\}. \tag{3.28}$$

According to the principle of optimality we know that regardless of the initial consumption decision, C_{T-2}, the remaining decision, C_{T-1}, must constitute an optimal consumption policy with respect to the financial assets, A_{T-1}, resulting from the first decision on consumption, C_{T-2}. Thus equation (3.28) can be written as

$$V_2^{opt}(A_{T-2}) = \max_{C_{T-2}} \left\{\phi(A_{T-2}, C_{T-2}) + V_1^{opt}(A_{T-1})\right\}. \tag{3.29}$$

In general, for an $(N-K)$-stage process at time t, i.e. for K period to go, where $K = (N-T+t)$, we have

$$V_{N-K}^{opt}(A_t) = \max_{C_t,C_{t+1},\cdots,C_{T-1}} \left\{\sum_{t=t}^{T-1} \phi(A_t, C_t) + \psi(A_T)\right\},$$

and as before, the application of the principle of optimality yields

$$V_{N-k}^{opt}(A_t) = \max_{C(t)} \left\{\phi(A_t, C_t) + V_{N-k-1}^{opt}[\phi(A_t, C_t)]\right\}. \tag{3.30}$$

Equation (3.30) is known as the basic *recurrence functional equation.* This equation generates the optimal return or optimal "cost-to-go" for an $(N-K)$-stage process if we know the optimal return for an $(N-K-1)$-stage process. To see this, let us assume that the set of all admissible financial assets and all admissible consumption policies are known. Using the recurrence functional equation (3.30) and applying all admissible consumption values to each admissible financial asset gives

$$V_0^{opt}(A_T) = \psi(A_T).$$

Equation (3.30) also implies that

$$V_1^{opt}(A_{T-1}) = \max_{C_{T-1}} \left\{\phi(A_{T-1}, C_{T-1}) + V_0^{opt}(A_T)\right\}.$$

From the available set of financial assets at time $T-1$, we select the first one and apply all the admissible consumption policies at time $T-1$ to compute the following: i) the next period financial asset A_T using the equation of motion and ii) $\phi(A_{T-1}, C_{T-1})$.

The A_T values obtained in (i) above enable us to choose the corresponding set of $V_0^{opt}(A_T)$. The following term

$$\phi(A_{T-1}, C_{T-1}) + V_0^{opt}(A_T)$$

determines a set of quantities of which the largest one is $V_1^{opt}(A_{T-1})$. The second, third, ... and nth values of the available set of financial assets will then be chosen. Each of these values together with the available set of C_{T-1} will determine the set of A_T, V_0^{opt} and $\phi(A_{T-1}, C_{T-1})$. The largest quantity of

$$\phi(A_{T-1}, C_{T-1}) + V_0^{opt}(A_T)$$

is $V_1^{opt}(A_{T-1})$. The associated values of A_{T-1} and C_{T-1} would be retained in the memory.

Equation (3.30) implies that

$$V_2^{opt}(A_{T-2}) = \max_{C_{T-2}} \left\{ \phi(A_{T-2}, C_{T-2}) + V_1^{opt}(Y_{T-1}) \right\}.$$

Analogously, if we select the first value of the available set of financial assets in period $T-2$ and try all the admissible consumption policies stored in the memory so far, we will obtain the followings: i) A_{T-1} by using our equation of motion and ii) $\phi(A_{T-2}, C_{T-2})$. We now store the value of A_{T-1} and examine the associated $V_1^{opt}(A_{T-1})$ which has already been stored. The largest quantity of

$$\phi(A_{T-2}, C_{T-2}) + V_1^{opt}(A_{T-1})$$

is $V_1^{opt}(A_{T-2})$. The associated C_{T-2} and A_{T-2} should be retained in the memory. If we select the second, third, ... and nth values of the available set of financial assets we can similarly produce a set of $V_2^{opt}(A_{T-2})$ which together with the associated A_{T-2} and C_{T-2} will be stored.

In period $t+1$, the information about $V_{N-K-1}^{opt}(A_{t+1})$ as well as the associated A_{t+1} and C_{t+1} are available. When we move back to period t, the available set of A_t and C_t will determine A_{t+1}. Compute $\phi(A_t, C_t)$ and choose $V_{N-K-1}^{opt}(A_{t+1})$ which is associated with the computed values of C_{t+1}. For each pair of financial asset and consumption policy at time t, we obtain

$$\phi(A_t, C_t) + V_{N-K-1}^{opt}(A_{t+1}),$$

the largest value of which is $V_{N-K-1}^{opt}(A_{t+1})$.

In period zero, the basic recurrence equation is

$$V_N^{opt}(A_0) = \max_{C_0} \left\{ \phi[A_0, C_0] + V_{N-1}^{opt}[\phi(A_0, C_0)] \right\}.$$

It is assumed that the initial level of consumer's financial asset A_0 is fully specified. Applications of all admissible consumption policies at time zero to the value of financial asset A_0 gives the followings: *i)* A_1 by using the equation of motion and *ii)* $\phi(A_0, C_0)$. As before, we compute the entire values of

$$\phi(A_0, C_0) + V_{N-1}^{opt}(A_1),$$

and choose the largest one as $V_N^{opt}(A_0)$. The associated policy value is the optimal consumption policy at time zero C_0^{opt}, which can be used to determine the optimal level of financial asset next period, i.e. A_1^{opt}, using the equation of motion. The stored information will give optimal value of the associated return and the associated consumption policies. C_1^{opt} will be used in equation of motion to produce A_2^{opt}, which provides, through inspection in the memory storage, the optimal consumption C_2^{opt} and optimal return at period 2, i.e. V_2^{opt}. We continue this process until all C_3^{opt}, $C_4^{opt}, \cdots$, C_{T-1}^{opt} and the resulting financial assets, i.e. A_3^{opt}, A_4^{opt},

$\cdots$, A_T^{opt} and their associated optimal return values are obtained. The optimal consumption trajectory is thus obtained.

3.4.2 Properties of the Optimal Consumption Path

In this section we derive the familiar Euler equation for optimal consumption by using the functional recurrence equation of dynamic programming together with the envelope relation. We will also show, following Blanchard and Fischer (1990), that Hall's random walk hypothesis can be directly derived by using the Euler equation thus obtained.

We start by defining the time-varying optimal value function $V_t(A_t)$ as

$$V_t(A_t) = maxE\left[\sum_{\tau=t}^{T}(1+\delta)^{-(\tau-t)}U(C_\tau)|\Omega_t\right], \tag{3.31}$$

subject to the equation of motion, i.e.

$$A_{t+1} = (1+r)A_t + Y_t - C_t.$$

All the terms are defined earlier. Equation (3.38) implies that the optimal value function is the present discounted value of expected utility evaluated along the optimal trajectory. For example,

$$V_t(A_t) = max\left\{E[U(C_t) + (1+\delta)^{-1}U(C_{t+1}) + \cdots + (1+\delta)^{-(T-t)}U(C_T)]|\Omega_t\right\},$$

or

$$V_{t+1}(A_{t+1}) =$$

$$max\left\{E[(1+\delta)^{-1}U(C_{t+1}) + (1+\delta)^{-2}U(C_{t+1}) + \cdots + (1+\delta)^{-(T-t)}U(C_T)]|\Omega_t\right\}.$$

Note that A_t is the only state variable in the model which is being directly affected by the only control variable in the model, i.e. the consumption C_t. Thus, the optimal value function is a function of the asset variable only. One can argue that the optimal value function is a function of conditional joint distribution of future labour income and rates of return. However, because our equation of motion for A_{t+1} does not allow for the impact of consumption on conditional joint distribution of future income (or the rate of return), these variables cannot be included as arguments in the optimal value function. The fact that the optimal value function is time variant captures the possibility of having different forms of this function over time.

Using equation (3.31), we write the functional recurrence equation as follows,

$$V_t(A_t) = \max_{C_t} \left\{ U(C_t) + (1+\delta)^{-1} E[V_{t+1}(A_{t+1})] | \Omega_t \right\}. \qquad (3.32)$$

Equation (3.32) is based on Bellman's principle of optimality stated earlier that "an optimal policy has the property that whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision". To maximise the right hand side of equation (3.32) subject to our equation of motion, we take the derivative of the right hand side with respect to C_t. This gives

$$\frac{\partial U(C_t)}{\partial C_t} + (1+\delta)^{-1} E \frac{\partial V_{t+1}(A_{t+1})}{\partial A_{t+1}} . \frac{\partial A_{t+1}}{\partial C_t} = 0.$$

Using the equation of motion to evaluate $\frac{\partial A_{t+1}}{\partial C_t}$, we find

$$U'(C_t) - (1+\delta)^{-1} E V'_{t+1}(A_{t+1}) = 0,$$

or

$$U'(C_t) = (1+\delta)^{-1} E V'_{t+1}(A_{t+1}). \qquad (3.33)$$

The functional form of the value function in equation (3.33), which is the first-order condition for optimality, is not known. We, therefore, cannot make any significant progress by using equation (3.33). However, we can use the envelope relation between $U'(C_t)$ and $V'(A_t)$ along the optimal trajectory. Consider a small variation in A_t in equation (3.32). We have

$$V'(A_t) = (1+\delta)^{-1} E \frac{\partial}{\partial A_t} \left[V_{t+1}(A_{t+1}) \right],$$

or

$$V'(A_t) = (1+\delta)^{-1} E \frac{\partial V_{t+1}(A_{t+1})}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial A_t},$$

which by using the equation of motion becomes

$$V'(A_t) = (1+\delta)^{-1} E V'_{t+1}(A_{t+1})(1+r).$$

Upon using equation (3.33) we have

$$V'(A_t) = (1+r)U'(C_t). \tag{3.34}$$

According to the envelope relation (3.34), the marginal value of financial assets along the optimal trajectory, is equal to the marginal utility of consumption multiplied by $(1+r)$. In other words, the marginal value of financial assets is equal to the increase in marginal utility of consumption at time $t+1$ viewed as of time t.

The appearance of $(1+r)$ in envelope relation (3.34) refers to the fact that the term C_t has entered the equation of motion at the end of period t while financial asset A_t is assumed to be known at the beginning of the period. Hence, A_{t+1} in the equation of motion, is a function of $(1+r)A_t$, whereas $Y_t - C_t$ is not multiplied by $(1+r)$. If we model the equation of motion as

$$A_{t+1} = (1+r)(A_t + Y_t - C_t), \tag{3.35}$$

the envelope relation would then appear as

$$V'(A_t) = U'(C_t). \tag{3.36}$$

Substituting the envelope relation (3.34) into the first-order conditions, equation (3.33), yields

$$U'(C_t) = (1+\delta)^{-1} E U'(C_{t+1})(1+r),$$

or

$$U'(C_t) = \frac{1+r}{1+\delta} EU'(C_{t+1}). \tag{3.37}$$

Equation (3.37) is the *Euler Equation* for the optimal consumption-saving problem. Note that changing the equation of motion to equation (3.35) and using the envelope relation (3.36), will not alter the Euler equation.

The Euler equation (3.37) is nothing but the generalization of Keynes-Ramsey condition under uncertainty; that is, the marginal rate of substitution between consumption in two periods is equal to the marginal rate of transformation. According to the Euler equation, if a consumer at time t reduces the consumption by ΔC and invests the resulting saving at the rate of interest r and consumes the proceeds at time $t+1$, then the decrease in utility at time t, which is $U'(C_t)$, must be equal to the increase in the expected utility at time $t+1$, i.e. $(1+r)E\,U'(C_{t+1})$, viewed as of time t, i.e. $(1+\delta)^{-1}[(1+r)E\,U'(C_{t+1})]$.

It is interesting to note that Hall's random walk hypothesis, which is based on life-cycle permanent income hypothesis, can be directly derived from the Euler equation which is based on the dynamic programming, [Blanchard and Fisher (1990)]. To see this, it suffices to write equation (3.37) as follows.

$$\frac{1+r}{1+\delta} U'(C_{t+1}) = U'(C_t) + \nu_{t+1},$$

where it is assumed that $E(\nu_{t+1}) = 0$. Alternatively,

$$U'(C_{t+1}) = \gamma U'(C_t) + \xi_{t+1}, \tag{3.38}$$

where $\gamma = \frac{1+\delta}{1+r}$. Moreover, under certain conditions, we can deduce from equation (3.38) that consumption follows a martingale. For example, assuming a quadratic utility function together with the assumption that the rate of interest r is equal to the rate of time preference δ, will result the followings: *i)* $U'(C_t)$ and $U'(C_{t+1})$

become linear functions in consumption and $ii)$ $\frac{1+\delta}{1+r}$ becomes 1 in equation (3.38). Thus, we can write

$$C_{t+1} = C_t + \xi^*_{t+1}, \tag{3.39}$$

where $\xi^*_{t+1} = k\xi_{t+1}$ and k is a constant.

3.4.3 The Bernoulli Case and Optimal Consumption Functions

In section 3.3.1 we used the method of maximum principle to obtain the optimal consumption-saving behaviour for an individual whose utility function is of the Bernoulli type. Here we solve the same problem using Bellman's dynamic programming. Since the utility function belongs to the class of those with hyperbolic absolute risk aversion, the application of the "guess technique" in solving the functional recurrence equation is possible. We have shown that such applications provide a relationship which specifies the optimal consumption as a non-linear function of consumer's time preferences and a linear function of wealth. This generalises the Blanchard and Fischer result (1990). Moreover, we have demonstrated how an explicit relationship for the value function, at the optimum level of consumption, can be achieved.

Consider the following dynamic optimization problem,

$$Max\, E\left[\sum_{t=0}^{T}(1+\delta)^{-t} lnC_t\right],$$

subject to

$$A_{t+1} = (1+r)A_t + Y_t - C_t,$$

where all the terms are defined earlier. Using equation (3.32) we write the functional recurrence equation as follows,

$$V_t(A_t) = \max_{C_t}\left\{lnC_t + (1+\delta)^{-1}E[V_{t+1}(A_{t+1})]|\Omega_t\right\}.$$

Assuming that the value function is time invariant, we obtain

$$V(A_t) = \max_{C_t} \left\{ lnC_t + (1+\delta)^{-1} E[V(A_{t+1})] | \Omega_t \right\}. \tag{3.40}$$

To obtain the property of optimal consumption trajectory, we differentiate the right hand side with respect to C_t to find

$$\frac{\partial lnC_t}{\partial C_t} + (1+\delta)^{-1} E \frac{\partial V(A_{t+1})}{\partial A_{t+1}} . \frac{\partial A_{t+1}}{\partial C_t} = 0,$$

or

$$\frac{1}{C_t} - (1+\delta)^{-1} E \frac{\partial V(A_{t+1})}{\partial A_{t+1}} = 0. \tag{3.41}$$

Since the utility function belongs to the class of hyperbolic absolute risk aversion, we can assume that the functional forms of the value function and the utility function are identical. This enables us to guess the following form for the value function whose validity will be checked at later stages. Our guess is therefore

$$V(A_t) = \alpha + \beta ln(A_t). \tag{3.42}$$

Substitute equation (3.42) into equation (3.41) to find

$$\frac{1}{C_t} - (1+\delta)^{-1} E \frac{\partial[\alpha + \beta ln(A_{t+1})]}{\partial A_{t+1}} = 0,$$

or

$$\frac{1}{C_t} - (1+\delta)^{-1} E \left(\frac{\beta}{A_{t+1}} \right) = 0.$$

We now substitute the equation of motion into the above equation. After rearrangement, we have

$$\frac{1}{C_t} = \frac{\beta}{(1+\delta)} . \frac{1}{(1+r)A_t + Y_t - C_t},$$

or

$$[\beta + (1+\delta)]C_t = (1+\delta)[(1+r)A_t + Y_t],$$

or

$$C_t = \frac{(1+\delta)}{\beta + (1+\delta)}[(1+r)A_t + Y_t]. \tag{3.43}$$

From the equation of motion we know that our individual consumer starts at period $(t+1)$ with the assets obtained from the previous period plus the accumulated interest, i.e. $(1+r)A_t$. When the individual receives Y_t, the sum of $(1+r)A_t + Y_t$ constitutes his wealth out of which the individual can consume. If we denote the above sum as A^* we have

$$C_t = \Phi A_t^*, \tag{3.44}$$

where

$$\Phi = \frac{1+\delta}{\beta + (1+\delta)}. \tag{3.45}$$

Equation (3.44) shows that the optimal consumption is a linear function of wealth. The optimal consumption clearly depends on β, the parameter of our initial guess for the value function $V(A_t)$. We can obtain the value of this parameter by substituting C_t from equation (3.44) into the functional recurrence equation (3.40), which also checks whether the initial guess has been correct. Note that if we obtain the value of β, an explicit relation for optimal consumption will then follow.

First substitute the initial guess into both sides of the functional recurrence equation. We have

$$\alpha + \beta ln A_t = \max_{C_t} \left\{ lnC_t + (1+\delta)^{-1} E[\alpha + \beta ln(A_{t+1})] \right\},$$

or

$$\alpha + \beta ln A_t = \max_{C_t} \left\{ lnC_t + \alpha(1+\delta)^{-1} + \beta(1+\delta)^{-1} E\, ln(A_{t+1}) \right\}.$$

Substitute the optimal consumption behaviour, equating (3.44), into the above equation to obtain

$$\alpha + \beta ln A_t = ln(\Phi A_t^*) + \alpha(1+\delta)^{-1} + \beta(1+\delta)^{-1} ln[(1+r)A_t + Y_t - C_t]. \tag{3.46}$$

The term $ln(...)$ in the third term on the right hand side of the above equation can be written as

$$ln[(1+r)A_t + Y_t - C_t] = ln(A_t^* - \Phi A_t^*) = ln(1-\Phi)A_t^*,$$

or

$$ln[(1+r)A_t + Y_t - C_t] = ln(1-\Phi) + lnA_t^*.$$

By substituting back into equation (3.46) we have

$$\alpha + \beta lnA_t = ln\Phi + \alpha(1+\delta)^{-1} + \beta(1-\delta)^{-1}ln(1-\Phi) + [1+\beta(1+\delta)^{-1}]lnA_t^*.$$

Since the functional form of the right hand side is exactly the same as that of the left hand side, we conclude that the initial guess was structurally correct. The corresponding parameters should, therefore, become equal to each other. This gives the following relations,

$$ln\Phi + \alpha(1+\delta)^{-1} + \beta(1-\delta)^{-1}ln(1-\Phi) = \alpha, \tag{3.47}$$

$$1 + \beta(1+\delta)^{-1} = \beta. \tag{3.48}$$

From equation (3.48) we have

$$\beta = \frac{1+\delta}{\delta}. \tag{3.49}$$

We now use the value of β from equation (3.49) to obtain Φ from equation (3.45). This gives

$$\Phi = (1+\delta) / \left[\frac{1+\delta}{\delta} + (1+\delta)\right],$$

or

$$\Phi = \frac{\delta}{1+\delta}. \tag{3.50}$$

The above value of Φ can be used to provide an explicit function for optimal consumption as follows,

$$C_t = \left(\frac{\delta}{1+\delta}\right) A_t^*. \tag{3.51}$$

We observe that the optimal consumption is a non-linear function of time preference and a linear function of wealth. The change in interest rates or in the future income affect optimal consumption in so far as they affect wealth.

The above procedure is quite general and the value of α can also be obtained alongside with the value of β from the system of equations (3.47) and (3.48). By substituting these values into the initial guess, i.e. $V(A_t) = \alpha + \beta ln(A_t)$, we can obtain an explicit relation for the value function.

3.5 Properties of the Optimal Consumption Path with Liquidity Constraints

We generalise the optimality condition [or the Euler equation (3.37)] to the case where consumption behaviour is constrained by liquidities. Using the method of dynamic programming, we show how the Lagrange multiplier measures the amount by which consumer's utility will change resulting from the relaxation of borrowing constraints. This section generalises section 3.3 to the case where liquidity constraints bind. Consider a consumer who wishes to maximise the following objective function

$$J = Max\, E\left[\sum_{t=0}^{T}(1+\delta)^{-t}U(C_t)\right],$$

subject to

$$A_{t+1} = (1+r)A_t + Y_t - C_t,$$

and the liquidity constraint

$$A_t \geq 0.$$

Following the argument we had in section 3.4.2, the functional recurrence equation is

$$V(A_t) = \max_{C_t}\left\{U(C_t) + (1+\delta)^{-1}E\,V(A_{t+1}) + \pi(A_{t+1})\right\}, \tag{3.52}$$

where π is the Lagrange multiplier for liquidity constraints. Recall that the notation λ has been reserved as the multiplier for the equation of motion. Differentiate the right hand side with respect to C_t to obtain

$$\frac{\partial U(C_t)}{\partial C_t} + (1+\delta)^{-1} E \left\{ \frac{\partial V(A_{t+1})}{\partial A_{t+1}} . \frac{\partial A_{t+1}}{\partial C_t} \right\} + \frac{\partial \pi(A_{t+1})}{\partial A_{t+1}} . \frac{\partial A_{t+1}}{\partial C_t} = 0,$$

or

$$\frac{\partial U(C_t)}{\partial C_t} - (1+\delta)^{-1} E \left\{ \frac{\partial V(A_{t+1})}{\partial A_{t+1}} \right\} - \pi = 0. \tag{3.53}$$

For the envelope relation, consider a small variation in A_t in equation (3.59),

$$V'(A_t) = (1+\delta)^{-1} E \frac{\partial}{\partial A_t} V(A_{t+1}) + \pi \frac{\partial (A_{t+1})}{\partial A_t},$$

or

$$V'(A_t) = (1+\delta)^{-1}(1+r) E\, V'(A_{t+1}) + (1+r)\pi. \tag{3.54}$$

If we now transfer the second and the third terms in equation (3.53) to the right hand side and then multiply both sides by $(1+r)$ we obtain exactly the right hand side of equation (3.54). We can then write

$$V'(A_t) = (1+r)U'(C),$$

or

$$V'(A_{t+1}) = (1+r)U'(C_{t+1}). \tag{3.55}$$

Substituting equation (3.55) into equation (3.53) gives

$$U'(C_t) = (1+\delta)^{-1}[(1+r) E\, U'(C_{t+1})] + \pi,$$

or

$$U'(C_t) = \frac{1+r}{1+\delta} E\, U'(C_{t-1}) + \pi. \tag{3.56}$$

Equation (3.56) is the optimality condition for consumption behaviour of a consumer with liquidity constraints. π which is the Lagrangian multiplier associated with the liquidity constraint measures the amount by which consumer's utility will change if current constraints on borrowing become relaxed by one unit. In other words, π represents the amount by which the marginal utility of borrowing will increase at period t by reducing the consumption next period. If π becomes zero, the liquidity constraint will be totally relaxed and the optimality condition (3.56) will be the same as in equation (3.37) where the marginal rate of substitution equals marginal rate of transformation.

3.6 The Generalised Hamiltonian, Liquidity Constraints and the Rejection of Hall's Random Walk Hypothesis

In section 3.5, we used Bellman's dynamic programming to derive the optimality conditions in consumption path when liquidity constraints were binding. In this section, we obtain the same result, i.e. equation (3.56), by applying the generalised Hamiltonian function in the maximum principle. This argument is the generalization of section 3.3 where the maximum principle was applied to the optimization of consumption behaviour without liquidity constraints.

Specifically, we answer the following two questions in this section: *i*) how can the existence of liquidity constraints directly influence the rate of change in the optimal consumption trajectory; and *ii*) how does Hall's random walk hypothesis of consumption collapse when an individual consumer is facing with the liquidity constraint. These two questions and particularly the latter, are of prime theoretical value since the rejection of orthogonality tests of Hall's random walk hypothesis is usually carried out on the alleged existence of liquidity constraints. Our results here constitute, for the first time, a strong theoretical framework for such empirical tests because it explicitly formulates the effects of liquidity constraints on optimal consumption path. We assume that consumers, in an imperfect capital market, face an upper limit to their net indebtedness which is a function of their income. The problem is to maximise the objective function, [equation (3.1)],

$$J = Max\, E \int_{t_0}^{T} e^{-\delta t} U[C(t)]\, d_t,$$

subject to the asset transition equation (3.2),

$$\dot{A}(t) = rA(t) + Y(t) - C(t),$$

and the assumption of liquidity constraints,

$$A(t) \geq -a - bY(t), \tag{3.57}$$

where a is the limit of net indebtedness.

Define the Hamiltonian as before, i.e. equation (3.6),

$$H = E\, U[C(t)]e^{-\delta t} + \lambda(t)[rA(t) + Y(t) - C(t)],$$

where $\lambda(t)$ is the adjoint variable. The control variable, $C(t)$, should maximise H subject to the inequality constraint (3.57). Writing equation (3.57) as $A(t) + a + bY(t) \geq 0$, we can construct the generalised Hamiltonian as follows,

$$H^* = E\, U[C(t)]e^{-\delta t} + \lambda(t)[rA(t) + Y(t) - C(t)] + \pi(t)[A(t) + a + bY(t)].$$

The control variable $C(t)$ can maximise H^* if

$$\frac{\partial H^*}{\partial C(t)} = 0,$$

or

$$E\, U'[C(t)]e^{-\delta t} = \lambda(t). \tag{3.58}$$

Equation (3.58) gives the optimal consumption $C(t)$ as a function of the adjoint variable $\lambda(t)$. To obtain the time path of the adjoint variable, we use the canonical equation, i.e.

$$-\frac{\partial H^*}{\partial A(t)} = \dot{\lambda}(t),$$

or

$$\dot{\lambda}(t) = -r\lambda(t) - \pi(t). \tag{3.59}$$

To obtain the properties of optimal consumption policy, we differentiate equation (3.58) with respect to time. Since $U'[C(t)]$ is a function of $C(t)$, we have

$$EU''[C(t)]E\frac{dC(t)}{dt}e^{-\delta t} - \delta e^{-\delta t}E\,U'[C(t)] = \dot{\lambda}(t),$$

and by using the equation for $\dot{\lambda}(t)$ we obtain

$$\{E\,U''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)]\}e^{-\delta t} = -r\lambda(t) - \pi(t).$$

Substitute equation (3.58) into the right hand side of the above equation and divide both sides by $e^{-\delta t}$, we have

$$E\,U''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)] = -rE\,U'[C(t)] - \pi(t)e^{\delta t},$$

or

$$\frac{E\,U''[C(t)]}{E\,U'[C(t)]}E\,\dot{C}(t) - \delta + r = -\frac{\pi(t)e^{\delta t}}{E\,U'[C(t)]}.$$

Assuming that

$$\psi(t) = \frac{\pi(t)e^{\delta t}}{E\,U'[C(t)]}, \tag{3.60}$$

and noting that $E\,\rho_a = -\frac{E\,U''[C(t)]}{E\,U'[C(t)]}$ is the coefficient of absolute risk aversion, we have $E\,\rho_a\dot{C}(t) = r - \delta + \psi(t)$. Assuming $\rho_a^* = E\,\rho_a$, then

$$\rho_a^* E\,\dot{C}(t) = r - \delta + \psi(t). \tag{3.61}$$

Equation (3.61), is an important result in consumption optimization when liquidity constraints bind. This equation implies that with a concave utility function, where $U''[C(t)]$ is negative and thus the coefficient of absolute risk aversion is positive, if liquidity constraints are binding at time t, i.e. $\psi(t) > 0$, then the optimal consumption increases, $\dot{C}(t) > 0$, provided that the interest rate is more than or equal to the subjective rate of time preferences, i.e. $r \geq \delta$. If liquidity constraints are not binding, $\psi(t) = 0$, then consumption will increase if $r > \delta$. However, equation (3.61) implies that with a concave utility function, the existence of liquidity constraints makes the optimal consumption to grow, not only when the interest rate is more than or equal to the subjective rate of time preference ($r \geq \delta$), but even when $r < \delta$, provided that $\psi(t) > r - \delta$. Under such conditions one can conclude that the liquidity constraint may shift the optimal consumption profile forward even when the rate of time preference exceeds the interest rate.

We now prove that the existence of liquidity constraints invalidates Hall's random walk hypothesis of optimal consumption. Recall that Hall's hypothesis has been frequently tested for the explanatory power of variables (other than consumption) in predicting consumption for the next period. Hall's hypothesis has been rejected since variables such as lagged stock prices or lagged income proved to be significant in explaining current consumption [see, for example, Hall and Mishkin(1982) and Zeldes(1989)]. It is well-known that the failure of random walk hypothesis of consumption is usually attributed to the presence of liquidity constraints, thus the existence of liquidity constraints and its impact on consumption have been tested indirectly. I have been unable to find, in the published literature on this subject, a complete theoretical treatment which shows *i*) how the existence of liquidity constraints can directly affect the optimal path of consumption and *ii*) how the random walk hypothesis of consumption breaks down when liquidity constraints are binding. I have answered the former question by deriving equation (3.61) and will answer the latter as follows.

Using the mean value theorem, we can write equation (3.61) as

$$E[C(t+h) - C(t)] = \frac{h[r - \delta + \psi(t)]}{\rho_a^*}.$$

By assuming $h = 1$ we have

$$C(t+1) = \frac{1}{\rho_a^*}(r - \delta) + C(t) + \frac{1}{\rho_a^*}\psi(t) + \xi_{t+1}. \tag{3.62}$$

Using Hall's condition for the random walk hypothesis, i.e. $r = \delta$, we have

$$C(t+1) = C(t) + \frac{1}{\rho^*}\psi(t) + \xi_{t+1}, \tag{3.63}$$

where

$$\psi(t) = \frac{\pi(t)e^{\delta t}}{E\,U'[C(t)]},$$

is defined in equation (3.60) and $\pi(t)$ is the adjoint variable associated with liquidity constraint in the generalised Hamiltonian function. Equation (3.62) implies that with a concave utility function, the expected value of optimal consumption increases when *i)* $r \geq \delta$ and *ii)* $\psi(t) > r - \delta$ even if $r < \delta$. Moreover, equation (3.63) indicates that Hall's random walk hypothesis can be rejected if liquidity constraints bind, i.e. $\psi(t) \neq 0$. This result holds even under the condition $r = \delta$.

3.7 Time-varying Interest Rates and the Properties of Optimal Consumption Path under Liquidity Constraints

As discussed before, the Euler equation approach in modelling consumer behaviour, which has been initiated by Lucas (1976) in his critique of standard estimation of consumption function, is based on the first order conditions in an individual's intertemporal optimization problem [equation(3.37)]. We have also noted that the rejection of the Euler equation,[71] has been related to the existence of liquidity constraints.[72] Note that the tests for the existence of liquidity constraints are usually carried out either indirectly, or are simply based on a particular assumed consumption-income relationship. Such problems in modelling the existence of liquidity constraints, within a rational expectations life-cycle permanent income hypothesis, are mainly rooted in the difficulties associated with finding satisfactory proxies for liquidity constraints. King (1986), Hayashi (1987)

[71]See, for example, Flavin(1981), Muellbauer (1983), Mankiw, Rotemberg, Summers (1985) and Campbell and Mankiw (1989, 1991).

[72]See, for example, Hall and Mishkin (1982), Hayashi (1985) and Zeldes (1989).

and Muellbauer and Latimore (1995) have reported the achievements in modelling the liquidity constraints in this direction.

It is possible to relax this theoretical shortcoming substantially by, first, introducing a function representing the structure of liquidity constraints (or the nature of capital market imperfections) and then accommodating this function within an individual's intertemporal optimization problem. This approach, which was adopted in the previous section, will be further developed here. We show how the generalised Hamiltonian function can be useful in modelling this problem.

In section 3.7.1 we will obtain an equation similar to equation (3.61) in which the change in consumption is related to the liquidity constraint, interest rates and consumer's time preferences. However, time-varying interest rates do not change our general conclusion which states that the existence of liquidity constraints necessarily results in an increasing consumption over time if the interest rate becomes equal to the subjective rate of time preferences. However, when $r(t) < \delta$, the consumption might not increase even if liquidity constraints bind. This, of course, depends entirely on the severity of the liquidity constraints: the condition $\psi(t) > r(t) - \delta$ might ensure a forward shift of optimal consumption profile when consumer's time preferences exceed the rate of interest. Using the Kuhn-Tucker condition, we will analyse, in section 3.7.2, the interactions between time-varying interest rates, the utility discount rate and the severity of liquidity constraints. This section generalises the results of Heller and Starr (1979) to the case where the interest rate is time-varying and the liquidity constraint specifies the consumer's net indebtedness as a function of income.

Using an inverse relationship between the coefficient of absolute risk aversion and the intertemporal elasticity of substitution, we obtain, in section 3.7.3, a relationship in which the intertemporal elasticity of substitution appears as the coefficient of liquidity constraint. This produces the important result that the response of optimal consumption to variations in the severity of liquidity constraints will be conditioned by the consumer's intertemporal elasticity of substitution.

3.7.1 Time-varying Interest Rates and Liquidity Constraints

This section generalises the results we obtained in section 3.6 by using a time-varying interest rate. In equation (3.61), we demonstrated the effects of liquidity constraints on the optimal time path of consumption; and equation (3.62) clearly showed how the existence of liquidity constraints could invalidate Hall's random walk hypothesis of consumption. We will examine these results under the condition of time-varying interest rates. The generalised Hamiltonian function will be used throughout.

The problem is to maximise the following objective function,

$$J = Max\, E \int_{t_0}^{T} e^{-\delta t} U[C(t)]\, dt,$$

subject to the asset transition equation,

$$\dot{A}(t) = r(t)A(t) + Y(t) - C(t), \tag{3.64}$$

and the following constraints on borrowing,

$$A(t) \geq -a - bY(t).$$

Note that the specification of the objective function and liquidity constraints are as before whereas the asset transition equation embodies a time-varying interest rate.

Defining the generalised Hamiltonian as

$$H^* = EU[C(t)]e^{-\delta t} + \lambda(t)[r(t)A(t) + Y(t) - C(t)] + \pi(t)[A(t) + a + bY(t)], \tag{3.65}$$

the optimal value of consumption will maximise H^* if

$$\frac{\partial H^*}{\partial C(t)} = 0,$$

or

$$EU'[C(t)]e^{-\delta t} = \lambda(t). \tag{3.66}$$

To obtain the time path of adjoint variable $\lambda(t)$, we use the canonical equation

$$-\frac{\partial H^*}{\partial A(t)} = \dot{\lambda}(t),$$

or

$$\dot{\lambda}(t) = -r(t)\lambda(t) - \pi(t). \tag{3.67}$$

Differentiating equation (3.66) with respect to time and substituting for $\dot{\lambda}(t)$ from equation (3.67) yields

$$EU''[C(t)]E\,\dot{C}(t) - \delta E\,U'[C(t)] = -r(t)E\,U'[C(t)] - \pi(t)e^{\delta t}.$$

Dividing both sides by $E\,U'[C(t)]$ and defining $\psi(t)$ as

$$\psi(t) = \frac{\pi(t)e^{\delta t}}{E\,U'[C(t)]},$$

yields

$$E\rho_a E\,\dot{C}(t) = r(t) - \delta + \psi(t),$$

or

$$\rho^* E\,\dot{C}(t) = r(t) - \delta + \psi(t), \tag{3.68}$$

where ρ_a is the coefficient of absolute risk aversion and $\rho_a^* = E\,\rho_a$.

Equation (3.68) is the generalization of equation (3.61) where the interest rate is not constant. According to equation (3.68), for any concave utility function, where $\rho_a > 0$, the existence of liquidity constraints, $\psi(t) > 0$, does not necessarily result in an increasing consumption over time. This is due to the variability of the interest rate. Despite structural similarities between equations (3.68) and

(3.61), one can argue that for periods in which $r(t) < \delta$, consumption might not increase due to the existence of liquidity constraints if $\psi(t) < \delta - r(t)$. Moreover, the condition $\psi(t) > r(t) - \delta$ ensures that optimal consumption profile will shift forward even if $r(t) < \delta$.

3.7.2 Liquidity Constraints and the Interactions between $r(t)$ and δ

In the previous section we proved that the effect of liquidity constraints on the pattern of optimal consumption over time depends on the relative magnitude of time varying rates of interest, the utility discount rate and the severity of liquidity constraints. In this section, we show that the Kuhn-Tucker conditions for optimization of consumer's utility over time can provide a useful relationship between these factors. Heller and Starr (1979) have used the Lagrangian approach to the same problem with time-invariant interest rates and a liquidity constraint in the form of non-negative assets, i.e. $A(t) \geq 0$. This section extends their analytical framework and generalises their results by introducing *i*) time-varying interest rates and *ii*) a liquidity constraint which specifies consumer's net indebtedness as a function of his income, i.e. $A(t) = -a - bY(t)$.

Consider an individual consumer maximising the following discrete utility function,

$$J = Max \sum_{t=0}^{T} (1+\delta)^{-t} U(C_t), \tag{3.69}$$

subject to the following asset transition equation

$$A_{t+1} = (1 + r_t)A_t + Y_t - C_t,$$

and the liquidity constraints,

$$A_t \geq -a - bY_t. \tag{3.70}$$

It is assumed that the consumer's initial asset is non-negative and is given, i.e.

$$A_0^* \geq 0.$$

The Lagrangian function for this problem is

$$L = \sum_{t=0}^{T} \left\{ (1+\delta)^{-t} U(C_t) + \lambda_t [(1+r_t)A_t + Y_t - C_t - A_{t+1}] + \pi_t [A_t + a + bY_t] \right\}. \tag{3.71}$$

The second term in equation (3.71) includes A_t and A_{t+1}. To avoid problems which might arise in differentiating L with respect to A_t, we can divide the planning horizon in the Lagrangian as follows,

$$\begin{aligned} L = {} & U(C_0) + \lambda_0[(1+r_0)A_0^* + Y_0 - C_0] + \pi_0[A_0^* + a + bY_0] + \\ & \sum_{t=1}^{T} \{(1+\delta)^{-t} U(C_t) + \lambda_t(Y_t - C_t) + [(1+r_t)\lambda_t - \lambda_{t-1}]A_t + \\ & \qquad \pi_t[A_t + a + bY_t]\}. \end{aligned}$$

The Kuhn-Tucker necessary conditions for an optimum are

$$\frac{\partial L(C_t, A_t, r_t, \lambda_t, \pi_t)}{\partial C_t} \leq 0, \quad C_t \geq 0 \quad \textit{with complementary slackness}, \tag{3.72}$$

$$\frac{\partial L}{\partial A_t} = 0, \tag{3.73}$$

$$\frac{\partial L}{\partial \lambda_t} \geq 0, \quad \lambda_t \geq 0 \quad \textit{with complementary slackness}, \tag{3.74}$$

$$\frac{\partial L}{\partial \pi_t} \geq 0, \quad \pi_t \geq 0 \quad \textit{with complementary slackness}. \tag{3.75}$$

By assuming $U'(0) = +\infty$, the condition $C_t > 0$ in equation (3.72) ensures that the left hand inequality binds, i.e.

$$(1+\delta)^{-t} U'(C_t) - \lambda_t = 0. \tag{3.76}$$

From equation (3.73) we have

$$(1+r_t)\lambda_t - \lambda_{t-1} + \pi_t = 0,$$

or

$$\lambda_{t-1} - (1 + r_t)\lambda_t = \pi_t. \tag{3.77}$$

In equation (3.74), $\frac{\partial L}{\partial \lambda}$ gives the asset transition equation which is an equality by definition. Therefore, $\frac{\partial L}{\partial \lambda}$ is binding and thus $\lambda(t)$ should be slack, i.e.

$$\lambda_t \geq 0. \tag{3.78}$$

By the same argument, equation (3.75) gives

$$A_t + a + bY_t \geq 0, \quad \pi_t \geq 0 \quad \textit{with complementary slackness.}$$

If our liquidity constraints bind, then we have

$$\pi_t > 0, \tag{3.79}$$

or

$$\pi_t(A_t + a + bY_t) = 0. \tag{3.80}$$

Equation (3.80) implies that the Kuhn-Tucker multiplier for liquidity constraint is non-zero if equation (3.70) holds, i.e. liquidity constraints bind. By equation (3.77) we have

$$\lambda_{t-1} \geq (1 + r_t)\lambda_t, \tag{3.81}$$

if π_t is nonnegative. According to equation (3.80), if liquidity constraints bind then (3.81) holds with strict inequality. From equation (3.76) we have

$$(1 + \delta)^{-(t-1)}U'(C_{t-1}) = \lambda_{t-1},$$

or alternatively,

$$U'(C_{t-1}) = (1 + \delta)^{t-1}\lambda_{t-1}, \tag{3.82}$$

which by substituting equation (3.81) for λ_{t-1} yields

$$U'(C_{t-1}) > (1 + \delta)^{t-1}(1 + r_t)\lambda_t. \tag{3.83}$$

From equation (3.76) we know that

$$U'(C_t) = (1+\delta)^t \lambda_t. \tag{3.84}$$

Substitute equation (3.84) into equation (3.83) to obtain

$$U'(C_{t-1}) > U'(C_t)(1+\delta)^{-1}(1+r_t),$$

or

$$U'(C_{t-1}) > \frac{1+r_t}{1+\delta} U'(C_t). \tag{3.85}$$

Equation (3.85), which is based on the assumption of binding liquidity constraint, is an important result. It clearly invalidates Hall's random walk hypothesis.

Recall that Hall's random walk hypothesis of consumption is based on the assumption of equality between interest rate and subjective time rate of discount. Equation (3.85) implies that even if $r_t = \delta$, we have

$$U'(C_{t-1}) > U'(C_t). \tag{3.86}$$

With regard to the assumed concavity of utility function, i.e. $U''(C_t) < 0$, equation (3.86) implies that consumption is increasing over time whenever the liquidity constraint is binding.

3.7.3 Interest Rates, Intertemporal Elasticity of Substitution, and Liquidity Constraints

It is well known that the effect of interest rate variations on consumption can best be analysed by intertemporal elasticity of substitution. The analytical framework to study the intertemporal elasticity of substitution in an optimal consumption programme with liquidity constraints are not yet satisfactorily developed. After a brief introduction to this problem, I will demonstrate how the optimal control theory can contribute towards formulating this problem by utilising the coefficient of absolute risk aversion.

An individual's plan to change his consumption from one period to the next depends, in principle, upon his expectations of real interest rate variations. An increase in the expected real interest rate tends to lower current consumption. The associated intertemporal substitution effect and particularly the response of consumption to a change in real interest expectation, which is captured by the intertemporal elasticity of substitution, plays an important role in determining the rate of change in consumer's future consumption.[73]

From a theoretical $RE-LC-PIH$ point of view, any change in the household's consumption plan depends on the expectations of real interest rates. Unfortunately, empirical work on intertemporal elasticity of substitution in consumption resulting from interest rate variations do not usually provide a consistent analytical framework to study the relationship between interest rate and consumption. The following are some examples.

Hall (1985) reports that despite fluctuations in the expected real interest rates (expected return from common stocks, treasury bills, and savings accounts), there has been only small shifts in the rate of growth of consumption. Using the US data, he concludes that the estimated values of intertemporal elasticity of substitution are small which supports the strong proposition that the elasticity is unlikely to be much above 0.1 and may well be zero. Hall's model of estimating the intertemporal elasticity of substitution has been extended by Campbell and Mankiw (1989, 1990) to take into account the possibility that some households behave as *rule of thumb* consumers in the sense that they do not obey the $LC-PIH$. Their estimates for the US data shows that about one-half of income accrues to such individuals due to the existence of liquidity constraints. Jappelli (1990) has estimated that such liquidity constrained consumers account for 18-19 per cent of all US consumers

[73]See Hall(1981), Mankiw, Rotemberg, and Summers (1985), Patterson and Pesaran (1992), among others.

who receive about 11-12 per cent of income.

More recently, Patterson and Pesaran (1992) have reported that, by using an instrumental variable moving average technique ($IVMA$), a re-estimation of Hall's model (1988) on quarterly data for the US economy provided an estimate of the intertemporal elasticity of substitution which is significantly different from zero. They concluded that there are strong indications of misspecification in the model used by Hall since he did not consider the non-$LC - PIH$ consumers; and when such consumers were taken into account the rate of change of income would become a significant explanatory variable. Moreover, they reported that Campbell and Mankiw's "one-half" result has proved not to be robust to changes in the sample period. Thus, upon some modifications, the proportion of income going to liquidity constrained consumers were found to be around 11 per cent.

Let us now present our formulation of interactions between interest rates, intertemporal elasticity of substitution and liquidity constrained consumption. Intuitively, the intertemporal elasticity of substitution can be negatively related to the coefficient of absolute risk aversion. Defining

$$\rho_a = -\frac{U''[C(t)]}{U'[C(t)]} = -\frac{d\, lnU'[C(t)]}{dC}$$

as the coefficient of absolute risk aversion ($CARA$), we know that $U''[C(t)]$ is a measure of concavity of the utility function. It is known that a consumer with a sharply concave utility function will, according to $LC - PIH$, avoid intertemporal substitution and will, therefore, prefer to smooth the consumption path over his planning horizon. Since $U''[C(t)] < 0$, such consumers will have high ρ_a which accompanies a low intertemporal elasticity of substitution. The same argument applies for constant relative risk aversion ρ_r defined as [see Seldon (1978)],

$$\rho_r = -\frac{U''[C(t)]}{U'[C(t)]}.C(t) = -\frac{d\, lnU'[C(t)]}{d\, lnC(t)}.$$

It should be noted that only for some class of utility functions is the intertemporal elasticity of substitution just equal to the reciprocal of the coefficient of absolute risk aversion [see Hall (1985)]. However as Hall (1988) reports, it seems that the best way to estimate the intertemporal elasticity of substitution is simply by regressing the log-change in consumption on the expected real interest rate because, intuitively, the rate of change in consumption over time can reveal the magnitude of the intertemporal elasticity of substitution in consumption.

In what follows, I derive an approximate relation between the intertemporal elasticity of substitution and the coefficient of absolute risk aversion. Using this relationship, I will then show how the time-varying interest rate and liquidity constraint can affect the changes in optimal consumption through the intertemporal elasticity of substitution. More specifically, we demonstrate that the consumer's intertemporal elasticity of substitution will condition the response of optimal consumption variations to liquidity constraints.

Defining the intertemporal elasticity of substitution as

$$\sigma = \partial \, ln \left\{ \frac{U'[C(t+1)]}{U'[C(t)]} \right\} / \partial \, ln \, r_t, \tag{3.87}$$

and expanding $U'[C(t+1)]$ by a Taylor series, we have

$$U'[C(t+1)] = U'[C(t)] + \Delta C(t+1)U''[C(t)] + \cdots.$$

Dividing both sides by $U'[C(t)]$ yields

$$\frac{U'[C(t+1)]}{U'[C(t)]} = 1 + \Delta C(t+1)\frac{U''[C(t)]}{U'[C(t)]}. \tag{3.88}$$

Note that $\frac{U''[C(t)]}{U'[C(t)]}$ in the second term on the right hand of equation (3.88) is exactly ρ_a with an opposite sign. Thus ρ_a will be negatively related to $\frac{U'[C(t+1)]}{U'[C(t)]}$ on the left hand side of this equation. This completes our proposition that the coefficient of absolute risk aversion and intertemporal elasticity of substitution are inversely related, i.e.

$$\rho_a = f^{-1}(\sigma). \tag{3.89}$$

We can now return to our familiar consumption optimization problem with a liquidity constraint of the form $A(t) \geq -a - bY(t)$ and an asset transition equation with time-varying interest rate. Recall equation (3.68), which was obtained by the application of Pontryagin's maximum principle, namely,

$$E\rho_a E\,\dot{C}(t) = r(t) - \delta + \psi(t),$$

where $\psi(t)$ was defined as

$$\psi(t) = \frac{\pi(t)e^{\delta t}}{E\,U'[C(t)]},$$

and $\pi(t)$ is the adjoint variable associated with the liquidity constraint. Substituting equation (3.89) into equation (3.68) gives

$$E\dot{C}(t) = f(\sigma)r(t) - f(\sigma)\delta + f(\sigma)\psi(t). \tag{3.90}$$

Equation (3.90) has important implications. It clearly specifies how the rate of interest affects the change in consumer's optimal consumption trajectory through the intertemporal elasticity of substitution as its coefficient. Moreover, it states that the intertemporal elasticity of substitution appears as the coefficient of the liquidity constraint. Equation (3.90) implies that the response of optimal consumption to liquidity constraints will be conditioned by the consumer's intertemporal elasticity of substitution. It is interesting to examine the term $f(\sigma)\psi(t)$. This term captures simultaneously the effects of *i*) pure preference parameters such as the utility function and the subjective rate of preference; *ii*) interest rate variations; and *iii*) structural parameters in the credit market which are manifested in the formulation of liquidity constraints. The first two factors are reflected by the intertemporal elasticity of substitution and the third factor is captured by $\psi(t)$. Moreover, according to equation (3.90), the time-varying interest rate affects the relationship between liquidity constraints and optimal consumption policies through its effects on the intertemporal elasticity of substitution.

3.8 Optimal Consumption in a Stochastic Environment

The discussion we had so far on deterministic dynamic choice-theoretic consumption models was based on the following assumptions: 1) The objective of choice was real consumption expenditures; 2) The utility function, representing preferences for the objective of choice, is monotonically increasing and strictly concave; 3) Constraints facing the agent can be summarised in a budget constraint; and 4) The agent's optimal choice maximises utility over the planning horizon subject to the assumed constraints. We have also used the following auxiliary assumptions: 1) The agent is assumed to be rational; 2) The agent's planning horizon is the lifetime of himself (and his spouse if applicable and not of his parents or mature children); 3) The agent has direct access to perfect capital and insurance markets; 4) The agent must be solvent at the end of the planning period; and 5) The intertemporal utility function is additive and depends on real consumption expenditures in each period. Moreover, these models assume that the following variables are certain: the length of the agent's lifetime or the planning horizon, the rate of return on investment, and the non-interest income which is also assumed to be exogenous. Different variants of the standard choice-theoretic consumption models are possible. An example is the existence of borrowing constraints which yields the liquidity constrained consumption models discussed in sections 3.6 and 3.7.

Within the above context, uncertain lifetime and uncertain future income are the two major sources which give rise to *stochastic optimal consumption behaviour.* Despite the importance of uncertain labour income in deciding on optimal consumption plans and despite a great deal of research work on life cycle and permanent income hypotheses, "yet, closed form decision rules for optimal consumption in the presence of uncertain labour income have not, in general, been derived. It seems strange that so much theoretical and empirical work has been done studying consumption and yet we do not even know what the optimal level of consumption or sensitivity of consumption to income should be in most very simple settings" (Zeldes, 1989, p. 275).

In this section, we consider the case where optimal stochastic consumption is constrained only by liquidity. Uncertainty about the longevity of life as well as uncertainty on the future income play the major roles in the stochastic behaviour of liquidity constrained consumers. It should be noted, however, that uncertain lifetimes can partially be responsible for uncertainty in future non-interest income simply because the latter is contingent on the survival of the agent and, therefore, becomes uncertain. In the following, we first examine the impact of uncertain lifetimes on optimal consumption behaviour. In section 3.8.2 we study this problem for uncertain income as well as uncertainty in the rate of interest. The implication of income uncertainty on optimal control applications to dynamic consumption choice is the subject of section 3.8.3.

3.8.1 Uncertain Lifetimes and the Optimal Consumption Behaviour

Consider an individual consumer maximising the following discrete utility function,

$$J = \max\, E_\tau \left[\sum_{i=t}^{\tau} (1+\delta)^{-i} U(C_i) \right], \tag{3.91}$$

where t is the current age and τ is the age that the consumer is expected to live. Clearly, τ is a stochastic integer in the range of $(t+1)$ and T, where T is the assumed (and certain) end of the planning period. Other terms are defined earlier. The individual consumer maximises (3.91) subject to the following asset transition equation,

$$A_{t+1} = (1+r_t)A_t + Y_t - C_t,$$

defined in equation (3.21) and the liquidity constraints

$$A_t \geq -a - bY_t,$$

defined in equation (3.70). It is also assumed that the consumer's initial asset is both non-negative and given.

Given survival to age t, let the probability of living to age $(t+1)$ be $\Pr L(t+1|t)$. The probability of living to age $(t+2)$ is $\Pr L(t+1|t) \Pr L(t+2|t+1)$. In general, the probability of living to age i, given survival to age t, is $\prod_{m=t+1}^{i} \Pr L(m|m-1)$. It follows, therefore, that the optimal consumption plan, $C^{opt}(t)$, which is contingent only on survival in each period, maximises the following objective function,

$$J = \sum_{i=t}^{T} \prod_{m=t+1}^{i} \Pr L(m|m-1)(1+\delta)^{-i} U(C_i). \tag{3.92}$$

The structure of equation (3.92) and the associated constraints, i.e. equations (3.28) and (3.82), is similar to the deterministic problem defined in sections 3.6.2 and 3.7.2, hence the results given there apply equally to the case of uncertain lifetimes.

The uncertain lifetime poses no problem in deriving tractable solutions to optimal consumption plans. To justify this conclusion, recall that the underlying variable in the uncertain lifetime is "survival", which can be regarded as a dichotomous dummy variable. The utility function, on the other hand, is non-zero if the survival variable is non-zero. This compatibility of the utility function and the survival variable ensures the similarity of objective function (3.92) and those discussed in sections 3.6.2 and 3.7.2 for deterministic case. The above similarity between stochastic and deterministic structures breaks down if the underlying variable in uncertain lifetime were continuous. For example, the assumption of dependency of utility derived from consumption on health invalidates the simple structure of the objective function (3.92). Similarly, for a multiple-person family, the uncertain lifetime renders the maximisation of equation (3.92) subject to the asset transition equation (3.21) and the liquidity constraints (3.82), inexpressible in a deterministic form presented in sections 3.6.2 and 3.7.2.

Multiple-person families pose serious problems for optimal consumption plans when lifetimes are uncertain. Optimal consumption in each period is contingent on family consumption in that period as well as on the probability distribution of future family consumption. The complicated mathematical expectations of family compositions at different periods makes it difficult to apply the method of dynamic programming to derive closed form solutions for optimal consumption decisions. An alternative approach is to assume that the family behaves as if expected family consumption in each future period will be realised with certainty. Mariger (1986) has used this approach to model the econometric specification of optimal consumption behaviour with uncertain lifetimes. The difficulty is to revise, at each period, the expectations of family composition in future periods in order to reflect the new information which has become available to the family. It follows that the optimal family's consumption plan cannot be projected without the knowledge of the time-paths of family composition since the optimal consumption plan is contingent on this composition.

3.8.2 Income Uncertainty and the Optimal Consumption Behaviour

Modelling stochastic future income in an individual's optimal consumption behaviour has always been a challenging issue. It was usually agreed that "it is not possible to obtain a closed-form solution for the optimal consumption plan when future labour income is uncertain" (Mariger, 1986, p. 59). The usual remedy was to eliminate all relevant income uncertainties by assuming a full insurance for net labour income in each period provided that at least one family member is alive. Note that there is no incentive for a single-person family to purchase such insurance.

The alternative practice was to assume that the individual consumer behaves *as if* expected future income is certain (see, for example, Hall and Mishkin, 1982). However, the seminal work of Dreze and Modigliani (1972) can be regarded as the most influential early contribution on optimal consumption decisions under uncertainty. It should be noted that Dreze and Modigliani's work benefited from the earlier work of Leland (1968) and Sandmo (1970) on optimal saving decisions under uncertainty. The classical work of Dreze and Modigliani considers an agent's optimal consumption decision in the context of two-period models. Mariger (1986) has extended their work to a multi-period model with additive intertemporal utility function. Following Dreze and Modigliani, consider an agent with a two-period horizon who receives a certain labour income YL_1 and an uncertain labour income YL_2 in periods one and two, respectively. The problem is to choose consumption in the first period, C_1, to maximise the following expected utility function which is assumed to be monotonically increasing and strictly concave,

$$J = E_{t=1}\{U[C_1, (1+r)(YL_1 - C_1) + YL_2]\}. \tag{3.93}$$

It is further assumed that $E_{t=1}(YL_2)$ equals the mean value of uncertain labour income in the second period, i.e. $\overline{YL_2}$. Let C_1^{opt} maximises equation (3.93) subject to asset transition equation and liquidity constraints and let $J^{opt}(C_1^{opt})$ be the associated cost. Now define $C_1^{+}(\alpha)$ as the optimal first period consumption when YL_2 is certain to be equal to α and let $J_1^{+}(\alpha)$ be the associated cost. Dreze and Modigliani show that

$$ARA(C_1^{opt}, C_2) \geq 0 \;\; or \;\; \leq 0 \quad for \;\; all \quad C_2 \rightarrow C_1^{opt} \leq \;\; or \;\; \geq C_1^{+}(YL_2^{+}), \tag{3.94}$$

where $ARA(C_1^{opt}, C_2)$ represents the change in the absolute risk aversion as C_1^{opt} increases and C_2 decreases along the budget line, YL_2^{+} is the certain level of income in period-two which makes the consumer as well off as with the uncertain income YL_2, that is

$$J_1^+[C_1^+(YL_2^+)] = J_1^{opt}(C_1^{opt}).$$

From equation (3.94) it follows that for decreasing absolute risk aversion, as C_2 increases and C_1 decreases along the budget line (with the expected utility held constant), the condition $ARA(C_1^{opt}, C_2) > 0$ provides a sufficient condition for saving, in the case of uncertainty, to become greater than in the certainty case. Moreover, since $YL_2^+ < \overline{YL_2}$ for a risk-averse consumer, then from equation (3.94) we have

$$ARA(C_1^{opt}, C_2) \geq 0 \quad for \ \ all \quad C_2 \to C_1^{opt} < C_1^+(\overline{YL_2}),$$

which means that decreasing absolute risk aversion ensures income uncertainty to increase saving above the level that would occur if YL_2 were certain to equal its mean, namely, $YL_2 = \overline{YL}$. In fact, higher variations of YL_2 around its mean, $\overline{YL}$, implies that the consumer will respond by saving more in the first period. In other words, current consumption is lower when labour income in each future period is stochastic than when it is certain to equal its mean. This implies that the assumption of certainty equivalence will tend to an overestimate of consumption particularly for working families who are heavily dependent on their labour income.

Capital income uncertainty, or more specifically, rate of return uncertainty, poses even more problems for dynamic optimal consumption decisions. The standard procedure is to take expectations over the portfolio rate of return with risky assets. The difficulty arises because the individual consumer must evaluate, in each period, the likelihood of becoming liquidity constrained in future periods when deciding on optimal current consumption. Simplifying assumptions can, of course, help towards obtaining a tractable solution. For example, it is usually assumed that the joint distribution of asset returns in each future period is known in the initial stage.[74]

Hakansson (1970) considers an agent facing risky assets whose rate of return are independently and identically distributed over time. He assumes that the intertemporal utility function is additive and is of the isoelastic form. Merton (1971) essentially considers the same problem for the continuous case where he assumes a Weiner process for asset returns. The assumption that the joint distributions of asset returns in each period are known in the initial period ensures that all the relevant past information is reflected in the current level of net worth. Hence, future wealth is the only uncertain variable in the model which is relevant to future consumption. Optimal consumption plan can thus be reduced to a consumption-investment plan contingent on wealth in each period. Note that the assumption of independently distributed capital returns plays a key role in this dynamic optimisation, since, otherwise, optimal consumption in each period would be contingent not only on wealth but on the sequence of past related rates of return on risky assets.

An important finding of Hakansson and Merton is that the demand for risky assets in each period is proportional to wealth, the proportion depending only on the joint distribution of asset returns. Moreover, the increased *capital risk* decreases (increases) the propensity to consume wealth in each period if η is less (greater) than zero. η is the exponent on consumption in an isoelastic utility function (used in Hakansson and Merton work) which measures the degree of concavity of the intertemporal utility function. A large value of η implies a larger consumption growth rate if the rate of interest exceeds the coefficient of time preference. It should be noted that "capital risk", in the above context, is defined according to Rothschild and Stiglitz (1971), namely, a capital asset is riskier if its

[74]See Hakansson (1970) and Merton (1971) for the early seminal work on optimal investment and consumption strategies under risk.

uncertain rate of return, r, is augmented to $r + \epsilon$, where ϵ is distributed as white noise.

An important point is that, Hakansson and Merton did not, unfortunately, pay much attention to the important role of liquidity constraints in their analysis of the impact of stochastic rate of return on optimal consumption plans. Recall that their major finding implies that the proportion of wealth invested in risky assets depends *only* on the joint distribution of asset return, that is, it is independent of the size and the composition of agent's wealth. It follows, therefore, that the individual consumer should borrow heavily in the early phases of the life cycle when his full wealth is mainly in the form of future earnings. Hence, an optimal portfolio choice can only be modelled correctly when liquidity constraints are fully taken into account. The converse is also true: an optimal consumption plan which takes liquidity constraints into account but does not acknowledge the effects of portfolio choice on risky assets will be equally inaccurate.

The findings of Hakansson and Merton, initially developed for a two-period model, were then generalised by Sibley (1975) and Miller (1976) for a multi-period optimal consumption plan with uncertain income. They showed that the results obtained by Hakansson and Merton are true for multi-period models. They also showed that beginning at any initial level of uncertainty on income, the precautionary saving increases with a Rothschild-Stiglitz (1970) mean-preserving spread on income. These results were further generalised by Zeldes (1984) to account for the sensitivity of consumption to wealth or to transitory income, i.e. the slope of the consumption function rather than its level. Using a second-order Taylor expansion of marginal utility, Zeldes showed that with constant relative risk aversion adding uncertainty raises the slope of the consumption function. This implies that income uncertainty makes consumption more sensitive to transitory income than under certainty equivalence. An interesting related result is that with constant absolute risk aversion the consumption function would shift downward in a parallel way when uncertainty is added, leaving the slope unchanged. In a similar development, Roel (1986) and Kimball (1988) have shown that excess sensitivity will occur for a class of utility functions that include constant relative risk aversion and excludes constant absolute risk aversion.[75]

Closed-form solutions for dynamic optimal consumption plans with stochastic income and constant absolute risk aversion have been reported by Schechtman and Escudero (1977), Caballero (1987) and Kimball and Mankiw (1987), among others. Zeldes (1989) is the first author who has successfully formulated an exact closed-loop solution for optimal consumption with uncertain income and a utility function which is constant relative risk aversion. Following Hall and Mishkin (1982), Zeldes assumes that income can be decomposed into two separate components. The first is the "lifetime" or permanent component which is assumed to follow a geometric random walk and is disturbed at each period by a random shock which captures the effects of pay rises, job changes, health changes and other similar persistent factors. The other component is the transitory component which is assumed to follow an $MA(2)$ and is hit each period by a random shock representing the effects of one-time bonuses, unemployment spells and other similar transitory factors. It is assumed that these two components are separately observable.

Zeldes (1989) has used the stochastic dynamic programming to calculate the optimal consumption plan with uncertain income. He formulated the problem simply as a stochastic control problem with only one state variable (wealth), one control variable (consumption) and one disturbance variable (income). Using a technique of Bertsekas (1976) in stochastic dynamic programming, Zeldes dis-

[75]See also Deaton (1991), Carroll (1992, 1994) and Flavin (1993).

cretized the state space into an S element grid. Beginning from the terminal period, a backward induction is used to solve for the value function and the corresponding optimal consumption. At each stage, the sum of current utility and the discounted expected value of next period's value function was maximised to yield the optimal level of consumption. It is well-known that the accuracy of the results depends upon the width of the grid used for the discretization in the stochastic dynamic programming framework. The approximate errors can thus be made arbitrarily small by narrowing the width of the grid at the expense of more computing time and excessive computer memory known as the *curse of dimensionality* in Bellman's dynamic programming.

However, Zeldes (1989) reports that the resulting consumption function is quite different from the certainty equivalence benchmark. The rational individuals with constant relative risk aversion develop an optimal consumption plan which "exhibits *excess* sensitivity to transitory income, hence they save *too* much and have expected growth of consumption that is *too* high relative to the simple permanent income hypothesis benchmark even in the absence of borrowing constraints" (pp. 295-296).

3.8.3 Implications of Income Uncertainty on the Applications of Optimal Control Theory to Dynamic Consumption Decisions

Departures from the certainty equivalence when an individual consumer is facing an uncertain future income pose serious problems in the application of optimal control theory to consumption optimisation. Recall that in the case of certainty equivalence, the optimal consumption is proportional to the sum of financial wealth and the present value of expected future income. As discussed in Chapter 2, the assumption of certainty equivalence significantly facilitates the applications of optimal control techniques to identify and compute optimal state and control trajectories.

As discussed in the previous section, the numerical complexities in calculating optimal consumption path when future income is uncertain -as is usually the case in practice- are beyond the capacity of human brain or the available business computers. This may render the standard optimisation of rational expectations permanent income/life cycle hypotheses unacceptable models of individual's consumption behaviour. This conclusion is based on Zeldes' results obtained by applying the direct search method in stochastic dynamic programming, which requires excessive computer memory beyond the capacity of the existing business computers. However, the application of alternative optimal control techniques, i.e. Pontryagin's maximum principle, is not promising either since obtaining a closed-form solution, when income is stochastic, requires complicated and heavily involved iteration techniques which similarly demands excessive computer storage capacities far beyond the human brain or the standard computers.

We are, therefore, faced with a very serious methodological problem in the formulation of dynamic intertemporal consumption decisions with stochastic income. The backward inductive procedure inherent in the stochastic dynamic programming plays the key role in this problem. The basis of the argument, which also constitutes a criterion for assessing the usefulness of an optimisation procedure, is best explained by Pemberton (1993, p. 3): "For the optimal solution to a model to be a useful guide to actual behaviour requires that the relevant agents in the real world can themselves identify and attain the solution (though not necessarily by using the same methods)". On the basis of this criterion, the standard stochastic dynamic programming fails to be accepted as a useful optimisation method in dynamic optimal consumption decisions because it is impossible to carry out the inherent backward inductive procedure. In fact, "either the problem gets so hideously complex that it is beyond the computational power of the decision-maker; or the sequence of implications stretches so far into the future that the consequences get shrouded in the mists of time" (Hay, 1983, p. 137).

Friedmanite defence of optimisation which heavily depends on "natural selection", "intuition" and "practice and/or learning" cannot save the backward induction procedure in computing dynamic optimisation of consumption from Pemberton's attack. Recall that in an optimal control of rational expectations permanent income/life cycle hypotheses, the optimal current consumption is contingent on future optimal decisions on consumption. As Pemberton (1993) argues, neither practice nor learning from past mistakes can make it easier to compute the optimal consumption sequence. Although the Euler equation, which is the necessary condition of optimisation, gives an intuitive meaning on balancing the marginal utilities of consumption in two periods, it does not provide any corresponding intuition for the actual consumption decisions. Moreover, "every individual consumer has to solve his or her own, unique lifetime backward induction problems and no *as if* simplifications are available" (Pemberton, 1993, p. 5). Thus, Friedman's natural selection argument and the associated concept of *innate abilities*, do not apply to the dynamic optimisation of consumer behaviour.

Despite Pemberton's effort to overcome this shortcoming by replacing the consumer's *detailed plan for the future* by their *concern for the future* and thus reducing the insoluble multi-period *backward induction* problem by a straightforward two-period *forward-looking* problem, much has to be done before a workable optimising model can be designed to explain and compute the individual's consumption behaviour in a stochastic environment. It appears that the following may facilitate the realisation of this objective.

i) A revival of research interest in "consumption function" approach (or solved-out consumption functions) to consumer behaviour may, in principle, avoid many computational problems associated with optimal control applications to dynamic optimisation of constrained consumption. For example, the curse of dimensionality in computing optimal consumption paths, reported by Zeldes (1989), can be avoided by adopting a solved-out consumption function approach since this does not depend on a backward induction procedure. The past 15 years have witnessed a flourishing literature on the Euler equation approach in permanent income-life cycle hypotheses. As Muellbauer and Lattimer (1995) point out "the enthusiasm of the Euler approach and the large scale abandonment of the solved out approach, particularly in North America, for modelling aggregate consumption, has gone too far" (p. 294).

ii) A new approach is required which brings together the theory-based Euler approach and the data-instigated "solved-out" consumption function approach to model the constrained optimal consumption decisions over time. This takes us back to the controversial issue of theoretical rigour against pure empiricism, i.e. theory against observation, discussed at length in Chapter two (section 2.5). The theoretical and empirical advances in optimal consumption behaviour and optimal consumption plans under income uncertainty (both labour and capital incomes uncertainties) during the past 20 years have revealed that the theoretical rigour of the Euler equation approach, which demands extreme and unrealistic assumptions about consumption behaviour, should somehow be relaxed at the expense of introducing more theoretical elaboration into the simple "solved out" rational expectations permanent income/life cycle hypotheses of consumption behaviour in a stochastic environment.

3.9 Summary and Concluding Results

After a brief introduction to the theoretical problems associated with liquidity constraints in households' consumption behaviour in section 3.2, our analysis of control theory applications to consumption behaviour starts with an examination of the properties of the optimal consumption path in the absence of liquidity constraints. This provides a basis for analyzing the effects of liquidity constraints on the optimal consumption path. It is well-known that the Euler equation approach in the consumer utility maximization reduces the multi-stage consumer dynamic optimization problem into a sequence of two-period decision process in which consumer's decision on next period's consumption depends on the current consumption given consumer's current income and financial assets. One can easily obtain the Euler equation using the Lagrange multiplier. I have shown in section 3.3 that Pontryagin's maximum principle and Bellman's dynamic programming can both directly be applied to consumer's multi-stage decision processes in order to obtain the optimality conditions for consumption growth path as well as deriving the Euler equation. Although the Euler equation and the optimality conditions thus obtained are exactly the same as those produced by the Lagrange method for a two period case, the methodology employed in solving the multi-stage decision process using the optimal control theory provides an opportunity for introducing different assumptions on consumer's utility specifications or on the constraints in the state and control space. My other related results are as follows.

i) Using a Bernoulli-type utility function and obtaining a differential equation for optimal consumption behaviour by applying the maximum principle, the following optimality condition for an increasing consumption path is obtained: the interest rate should exceed the consumer's rate of time preference (section 3.3.2).

ii) The application of Bellman's dynamic programming to a problem similar to (i) above produces exactly the same result (section 3.4.3).

iii) Again, by using a Bernoulli-type utility function and by solving the functional recurrence equation in the dynamic programming, together with using the guess technique, we have demonstrated in section 3.4.3 that the optimal consumption is a non-linear function of the time preference and a linear function of wealth. The change in the interest rate or in the future income affect optimal consumption in so far as they affect wealth. This generalises the result of Blanchard and Fischer (1990).

Recall that specific solutions exist for a special class of consumption problems whose utility functions are hyperbolic absolute risk aversion (including the isoelastic or constant relative risk aversion and the quadratic utility functions). The solution to other classes of consumption problems are basically possible only by the direct search method using the recurrence equation in the dynamic programming which is itself based on Bellman's principle of optimality. To demonstrate the methodology of solving the recurrence equation backward in time to obtain an optimal closed-loop policy sequence for consumption, I have used in section 3.4.1 a general non-linear budget constraint and a non-linear value function. This appears to be the first attempt at demonstrating the potentiality of this control technique in dealing with non-linear economic dynamics and general objective functions.

The other important result obtained in this chapter on the optimization of consumption behaviour without liquidity constraints is the direct application of the dynamic programming together with the envelope theorem in order to derive the Euler equation for optimal consumption (section 3.4.2). Moreover, it is shown how this result will lead to Hall's random walk hypothesis for consumption.

In section 3.3.3, Pontryagin's maximum principle has been directly applied, for the first time, to the consumer's optimal consumption programme in order to derive Hall's random walk hypothesis of consumption behaviour. The utilisation of the coefficient of absolute risk aversion significantly facilitated this derivation. The result thus obtained is exactly the same as those derived by applying Bellman's dynamic programming. This completes the application of standard optimal control techniques in producing Hall's hypothesis which currently plays a vital role in the Euler approach to consumption theory.

By examination of the properties of the optimal consumption path when liquidity constraints are binding, I have produced a number of results, with and without time-varying interest rates. These results, which are presented in sections 3.5, 3.6 and 3.7, include the following:

i) By using the method of dynamic programming together with the envelope theorem, I obtained the optimality condition in terms of the Lagrange multiplier associated with the liquidity constraints. This multiplier represents the amount by which the consumer's utility will change if current constraints on borrowing become relaxed by one unit.

ii) By using the generalised Hamiltonian function in the maximum principle, I have shown how liquidity constraints can directly affect consumption behaviour along the optimal path.

iii) Again, by using the generalised Hamiltonian, I have shown how the existence of liquidity constraints rejects Hall's random walk hypothesis. We know that the effects of liquidity constraints on consumption are usually tested indirectly. Our result here is of prime theoretical value because, for the first time, it gives an explicit relationship between liquidity constraints and the random walk hypothesis.

iv) The above result has been obtained under the conventional assumption that an individual's net indebtedness is a constant function of his income. By using the generalised Hamiltonian, I have shown how different formulations of liquidity constraints can, in principle, be handled in dynamic optimization problems of consumer choice.

v) I have shown, using the Hamiltonian approach, how time-varying interest rates can affect consumption variations along the optimal consumption trajectory.

vi) Using the method of Kuhn-Tucker conditions, I have obtained an explicit relation which demonstrates how the existence of liquidity constraints can reject Hall's random walk hypothesis. The same result has been obtained earlier in this chapter by using the generalised Hamiltonian function. The application of the Kuhn-Tucker conditions, however, provides a better insight into the possible interactions between time-varying interest rates and the utility discount rate. However, the generalised Hamiltonian function has much wider capabilities in treating different models of liquidity constraints.

vii) I have obtained an approximate negative relation between the coefficient of absolute risk aversion and the intertemporal elasticity of substitution. I used this relation to generalise the above results further. The results obtained are very useful since they specify the following:

1. How the time-varying interest rates affect optimal consumption through intertemporal elasticity of substitution which acts as a coefficient.

2. How the intertemporal elasticity of substitution, which has appeared as the coefficient of liquidity constraint, affects the optimal consumption behaviour. It should be noted that equation (3.90) has an interesting property: it simultaneously captures the effects of the following variables on the optimal consumption path: *i*) the pure preference parameters; *ii*) the interest rates variations; and *iii*) the structural parameters prevailing in the credit markets which are manifested in the modelling of liquidity constraints.

Our analysis of the optimal consumption behaviour in a stochastic environment (section 3.8) shows that the departure from the certainty equivalence, when an individual consumer is facing an uncertain future income, poses serious problems in the application of optimal control theory to dynamic optimisation of consumption. Numerical complexities in calculating the optimal consumption paths with future uncertainties are beyond the capacities of either human brain or of the existing business computers. Hence, the backward induction procedure inherent in stochastic dynamic programming, as well as the iterative techniques associated with the maximum principle, may be unacceptable optimisation methods because the relevant agent in the real world cannot identify and attain the solution. We have concluded that a revival of research interest in the "solved-out" consumption functions and an attempt to bring this data-instigated approach closer to the theory-based Euler approach are needed to advance our understanding of optimal consumption behaviour.

Chapter Four

Optimal Control of Dynamic Leontief Models

4.1 Introduction

Recall that in an input-output model, the output at time t can be considered as a function of the capacity brought about by investment in the past. Investment at time t and subsequent years, will generate the greater capacity which the growing output will require in the future. It is the nature of such a link between the past and the future that allows for optimality in an input-output model. Resources, instead of being consumed now, can be devoted to investment in order to expand productive capacities for subsequent production, or commodities demanded in future can be produced now and stored. A choice must always be made at every stage if an *optimal* input-output planning model is desired. This provides the possibility of applying optimal control theory to input-output models.

Despite extensive use of input-output models in economic planning since its development in 1941 by Leontief, not much attention has been given to formulate this model as a problem in a dynamic optimization framework. The few writers who contributed to input-output models from a control theoretic point of view [for example Brody (1970) and Smirnov (1970)] have used either very simple models or have confined their analysis to deterministic models only. In this chapter, I have, for the first time, formulated and solved both deterministic and stochastic dynamic Leontief models with and without a substitution system and with a quadratic as well as non-quadratic (non-linear) objective functions. More important, the Leontief substitution system is formulated as a control process in which optimal substitute activities are derived by using Bellman's dynamic programming. The stochastic dynamic Leontief model with uncertainties about future values of output is then solved together with a stochastic substitution system which includes measurement errors.

Section 4.2 presents a brief discussion on the importance of the dynamic Leontief model in economic modelling from an historical perspective. The significance of treating Leontief's input-output model as an optimisation problem is also discussed. The deterministic dynamic Leontief model is formulated as a control problem in section 4.3. Section 4.4 deals with optimal consumption policies for dynamic Leontief's input-output model using Pontryagin's maximum principle. Using the Hamiltonian function, we have derived the matrix Riccati equation, which together with boundary conditions and tracking equation, provides the optimal sequence of consumption and the associated output vectors.

The analysis of optimal consumption policies for the dynamic Leontief model using Bellman's dynamic programming is the subject of section 4.5. In section 4.6 we have formulated the optimal control of input-output models with a substitution system in which an optimal policy sequence is derived which generates optimal substitute activities such that a cost function is minimised. The dynamic programming solution of the Leontief substitution system is the subject of section 4.7 in which Åstrom's fundamental lemma of stochastic control theory is used to obtain the functional recurrence equation for the stochastic non-linear Leontief substitution system with measurement error and uncertainties about the future state of the system. Again, the optimal substitute activities are derived with respect to minimization of a cost function.

Section 4.8 is concerned with optimal stochastic control of a dynamic Leontief model with future uncertainties and stochastic substitution. The method of stochastic dynamic programming is utilised to obtain the optimal consumption policies. The functional recurrence equation for the stochastic substitution system generates the sequence of optimal base matrices which are then used in the process designed to generate the optimal stochastic control of dynamic Leontief models. A feedback optimal control law of optimal sequence of final consumption and the associated optimal output trajectory are then obtained. And finally, section 4.9 provides a summary of the main results obtained in this chapter.

4.2 The Leontief Model and Mathematical Economic Modelling: Background, Importance and the Optimal Control Approach

Using an historical approach, we discussed in Chapter One (section 1.4), the controversial issue that whether economic truths are discoverable through the instrumentality of mathematics. We concluded that there are economic questions which can only be answered by specific mathematical methods.[76] Again, in Chapter One (section 1.7), we identified the conditions under which mathematization of economics can lead to results of significant value in analysing the real world economic issues. We showed that the applications of mathematical methods to economic analysis are most promising in those areas where the *abstract* economic concepts, to be used in mathematical machinery, are close approximations to economic realities. Inter-industry models, for which Leontief invented the input-output technique, is a remarkable example of successful applications of mathematics to economic analysis. An interesting point of theoretical importance is that in all these cases, the application of mathematics of higher levels have usually produced results of higher values. Let us examine this point for the Leontief

[76] See section 1.4.2.

input-output model in an historical perspective.

It is now agreed that the input-output analysis, as a theoretical construct, was known to economists well before 1933 when Leontief began seriously to develop his input- output technique. Perhaps, *Tableau Économique* can be regarded as the first attempt in which François Quesnay (1759) tried to depict the relationship between agriculture and other sectors.[77] Using elementary mathematics, he produced a simple version of an input-output model representing the functioning of the real world economy.

Nearly one hundred years later, Marx demonstrated, using a two-sector table, the relationship between the capital goods and consumer's goods sectors (or departments) in an economy. An examination of Marx's discussion on *Accumulation and Reproduction on an Extended Scale*,[78] suggests a strong theoretical similarity between Marx's schematic presentation of accumulation and circulation and the Leontief's input-output model. Some authors have even concluded that "the input-output calculations is simply an enlargement of the conditions of equilibrium in Marx's formulae of expanded reproduction. The two departments have been replaced by fourteen, which complicates the picture but makes it possible to bring it closer to reality" [Ernest Mandel (1962), p. 641]. It should be noted, however, that the main difference between Marx's table of expanded reproduction and Leontief's input-output analysis is that the focus of the latter is to calculate the intermediate commodities bought and sold among sectors, without any reference to the "value added" expressed by the number of hours of labour, whereas the

[77]François Quesnay (1694-1774) wrote the first edition of his *Tableau Économique* in 1758. A limited new edition of the book was published in 1759 at Versailles, and was presented to Louis XV. He published a simplified version of the *Tableau* in 1766 in an article which appeared in the *Journal de l'agriculture* with the title *Analyse de la formule arithmétique du Tableau économique*.

[78]See, *Capital, A Critique of Political Economy*, vol. II, Book II: The Process of Circulation of Capital, edited by F. Engels, pp. 489-523, particularly, the section entitled "Schematic Presentation of Accumulation", pp. 505-523, including the first and second illustrations.

former concentrates on calculating the total value of all the commodities bought and sold.

The real contribution of input-output approach in modelling the real economy emerged when Leontief started to quantify the Walras-Cassel formulation of the general equilibrium, which was popularised in the 1920's and 1930. We discussed in Chapter One (section 1.2.5) that, in his general equilibrium analysis, Walras has used the abstract theoretical concepts which strongly impeded the quantification of the relationships among the sectors of an actual economy. The celebrated work of Leontief (1941) is essentially the empirical groundwork of Walras's general equilibrium model, for which Leontief has used the U.S. economy as an example. This explains why Leontief's path-breaking book, i.e. *The Structure of American Economy* has been subtitled "An Empirical Application of Equilibrium Analysis".

Leontief's operational point of view led him to take the position that the main task of an economist is to show that the theory can be applied to real economies and that it leads not only to predictions about the future behaviour but the accuracy of predictions can also be tested. Hence, estimating the technical coefficients, which measure the relationships among different sectors and thus facilitate quantifying the effects of different economic policy regimes, became the central point in the input-output analysis.

Applying higher-level mathematics was the key to Leontief success. Computations with input-output models require inverting large matrices. The history of the developments in input-output models shows how the advances in mathematical methods (for example, the partitioning methods in matrix inversion) and the progress in computer technologies have significantly contributed towards solving the interesting real economic problems. Recall that when Leontief started his work on input-output models in 1933-1934, the punch-card computers could multiply, but could not divide and yet Leontief was working on a model of 44 simultaneous equations with about 2000 coefficients.

Despite all the merits and achievements, the dynamic Leontief model soon encountered serious theoretical shortcomings. The problem of causal indeterminacy, i.e. the possibility of the output and the stock of at least one commodity becoming negative for sufficiently large t, is one of the most notable difficulties associated with dynamic Leontief models.

The general instability characteristic of the dynamic Leontief model is mainly due to the model being strictly linear and deterministic. The elements of intermediate and capital coefficient matrices are fixed and there is no possibility of factor substitution in the production function, hence the price mechanism does not reflect the effects of demand and supply on the factors of production. In this case, the dynamic Leontief model can be regarded as a special case of the von Neumann model, in which there is only one production process for producing each good, with no joint output. With such constraints, unless the initial output and capital stock vectors are on a certain ray from the origin, there is no guarantee that the model will be stable.

The problem of instability of a dynamic Leontief model can partially be overcome once it is regarded as an optimal planning model with constraints imposed on output and capital stocks. A further development in this direction is to introduce a feedback mechanism in which the output of the system, at any period, is observed and used as an input to a controller process, which accordingly determines the optimal decision for the next period. This is the problem of optimal planning and control of a dynamic Leontief model, which constitutes an example of how the advances in mathematical methods (for example, optimal control theory) can be used to relax the theoretical shortcoming associated with theorisation of economic behaviour. In the next section, we formulate the dynamic Leontief model as a control problem and the subsequent sections consider the complete solutions to deterministic and stochastic cases. My main contribution in this chapter is first, to formulate and solve the Leontief substitution subsystem as a control problem and then, to obtain the optimal consumption sequence for the dynamic Leontief model (for both deterministic and stochastic cases) incorporating deterministic and stochastic substitute activities.

4.3 The Dynamic Leontief Model as a Control Problem

Let us define q_1 and q_2 the total current outputs, V_1 and V_2 the total stocks of capital in the form of commodities 1 and 2, I_1 and I_2 the investments defined as follows,

$$I_1(t) = V_1(t+1) - V_1(t),$$

$$I_2(t) = V_2(t+1) - V_2(t),$$

Γ the capital input-output coefficient matrix whose elements γ measure the required inputs of capital equipments to produce one unit of output, e the consumption vector and A the current input-output coefficient matrix whose elements measure the inputs of intermediate product needed to produce one unit of each product. The following inequalities will then indicate how the net output is limited by the pre-existing stocks of capital,

$$\gamma_{11}[I_1(t) + e_1(t)] + \gamma_{12}[I_2(t) + e_2(t)] \leq V_1(t), \tag{4.1}$$

$$\gamma_{12}[I_1(t) + e_1(t)] + \gamma_{22}[I_2(t) + e_2(t)] \leq V_2(t). \tag{4.2}$$

Equations (4.1) and (4.2) imply that the only limitation to the system's expansion is the scarcity of capital stock.

In general, the output of commodity i at time t, $q_i(t)$, can be used for three different purposes: $i)$ as current flow of intermediate products $Aq_i(t)$, $ii)$ as net addition to the capital stock $I_i(t)$, or $iii)$ as final consumption $e_i(t)$,

$$q_i(t) = Aq_i(t) + I_i(t) + e_i(t). \tag{4.3}$$

The productive capacity at time $t = 1$ will be determined by the given $V_1(0)$ and $V_2(0)$. The subsequent productive capacities depend upon the allocation made between consumption and addition to capital stock at each stage of the planning. An *optimal* feasible investment programme is thus associated with a unique feasible consumption pattern for the whole planning period, t=1, 2, ..., T.[79] The behaviour of an input-output model, therefore, depends upon the way of deciding the consumption pattern which involves a problem in the field of optimal control theory.

The level of final demand as a control variable in an input-output model requires the government, as the main economic controller, to affect final demand by its own current expenditures as well as through taxation and subsidies. This enables the government to realize the optimal capital stock trajectory necessary for optimal economic growth.

It is assumed that the state of an input-output model at time t is given by its gross production vector $q(t)$, which is also assumed to be exhausted by intermediate uses $Aq(t)$ and extensions to capital stock $I(t)$ and consumption $e(t)$, i.e.

$$q(t) = Aq(t) + I(t) + e(t). \tag{4.4}$$

The dynamics of investment or extension to productive capacity is given by the

[79] A feasible investment programme is defined as the one which satisfies inequalities (4.1) and (4.2) and similarly, a feasible consumption programme must satisfy restrictions on minimum or exponential minimum consumption.

following equation,

$$I(t) = \Gamma \Delta q = \Gamma[q(t+1) - q(t)], \tag{4.5}$$

where Δ is the first-difference operator. Substituting equation (4.5) into equation (4.4) yields

$$q(t+1) = \Gamma^{-1}(I - A + \Gamma)q(t) - \Gamma^{-1}e(t),$$

and by defining $\Gamma^{-1}(I - A + \Gamma) = B$, we have

$$q(t+1) = Bq(t) - \Gamma^{-1}e(t). \tag{4.6}$$

Equation (4.6) is the *system dynamic equation* for the Leontief dynamic model where $e(t)$ is the control vector and $q(t)$ is the state vector. It is also assumed that the state of the system at the initial stage $t = 0$ is given as $q(0)$.

We will study the dynamic input-output model as a tracking (or servo-mechanism) control problem in which we wish to track a desired output trajectory. The performance measure (objective or cost functions) for such a system is usually given as

$$W = 1/2 \sum_{t=0}^{T} \left\{ [q(t) - q^d(t)]' Q [q(t) - q^d(t)] + [e(t) - e^d(t)]' R [e(t) - e^d(t)] \right\}, \tag{4.7}$$

where $q^d(t)$ and $e^d(t)$ are the desired or nominal vectors of output and consumption trajectories to be tracked. Q and R are usually diagonal weighting matrices giving the relative *costs* of deviating from the desired trajectories. Q and R matrices may depend on time.

The deterministic optimal control problem for the dynamic Leontief model is to find an admissible final consumption sequence $e^{opt}(t), t = 0, 1, 2, \cdots, T-1$ satisfying the system dynamic equation (4.6) and in so doing minimising the cost function (4.7).

4.4 Optimal Consumption Policies for Dynamic Leontief Models using Pontryagin's Maximum Principle

We assume the following quadratic objective function for the dynamic Leontief model,

$$W = 1/2[q(T) - q^d(T)]'S[q(T) - q^d(T)] +$$
$$1/2 \sum_{t=0}^{T-1} \left\{[q(t) - q^d(t)]'Q[q(t) - q^d(t)] + [e(t) - e^d(t)]'R[e(t) - e^d(t)]\right\}. \qquad (4.8)$$

The above objective function gives particular attention to the constraints imposed on the system at the terminal stage. Adjoining the Leontief dynamic system to the objective function by using the Lagrange multiplier $\lambda(t)$, yields

$$W = 1/2[q(T) - q^d(T)]'S[q(T) - q^d(T)] +$$
$$1/2 \sum_{t=0}^{T-1} \left\{[q(t) - q^d(t)]'Q[q(t) - q^d(t)] + [e(t) - e^d(t)]'R[e(t) - e^d(t)]\right\} +$$
$$\lambda'(t+1)[Bq(t) - \Gamma^{-1}e(t) - q(t+1)], \qquad (4.9)$$

and introducing the Hamiltonian scalar function, we obtain

$$H[q(t), e(t), \lambda(t+1), t] = 1/2[q(t) - q^d(t)]'Q[q(t) - q^d(t)] +$$
$$1/2[e(t) - e^d(t)]'R[e(t) - e^d(t)] + \lambda'(t+1)[Bq(t) - \Gamma^{-1}e(t)]. \qquad (4.10)$$

Substituting equation (4.10) into equation (4.9) yields

$$W = 1/2[q(T) - q^d(T)]'S[q(T) - q^d(T)] + \sum_{t=0}^{T-1}[H - \lambda'(t+1)q(t+1)]. \qquad (4.11)$$

The problem now is to minimise equation (4.11). Using the method of perturbation and assuming that the correct optimum values of $q(t)$ and $e(t)$ as $q^{opt}(t)$ and $e^{opt}(t)$, we have

$$q(t) = q^{opt}(t) + \xi(t), \qquad (4.12)$$

$$q(t+1) = q^{opt}(t+1) + \xi(t+1), \qquad (4.13)$$

$$e(t) = e^{opt}(t) + \epsilon(t), \tag{4.14}$$

where $\xi(t)$ and $\epsilon(t)$ are variations in $q(t)$ and $e(t)$, respectively. Substituting equations (4.12-4.14) into equation (4.11) and performing the minimization, we obtain Pontryagin's necessary conditions for optimality, i.e. $\frac{\partial H}{\partial e(t)} = 0$, the coupling relation, or

$$R[e(t) - e^d(t)] - \Gamma'^{-1}\lambda(t+1) = 0, \tag{4.15}$$

and $\frac{\partial H}{\partial q(t)} = \lambda(t+1) - \lambda(t)$, the adjoint equation, or

$$Q[q(t) - q^d(t)] + (B' - I)\lambda(t+1) = -\lambda(t), \tag{4.16}$$

with the following boundary conditions

$$q(0) = given, \tag{4.17}$$

$$\lambda(T) = S[q(T) - q^d(T)]. \tag{4.18}$$

From equation (4.15) we have

$$e(t) = R^{-1}[\Gamma'^{-1}\lambda(t+1)] + e^d(t). \tag{4.19}$$

Substituting equation (4.19) into equation (4.6) results in

$$q(t+1) = Bq(t) - \Gamma^{-1}R^{-1}[\Gamma'^{-1}\lambda(t+1)] - \Gamma^{-1}e^d(t). \tag{4.20}$$

We now assume

$$\lambda(t) = \Phi(t)q(t) + \Theta(t). \tag{4.21}$$

Substituting equation (4.21) into equations (4.20) and (4.16) we obtain

$$q(t+1) = Bq(t) - \Gamma^{-1}R^{-1}\Gamma'^{-1}[\Phi(t+1)q(t+1) + \Theta(t+1)] - \Gamma^{-1}e^d(t), \tag{4.22}$$

and

$$-\Phi(t)q(t) - \Theta(t) = Q[q(t) - q^d(t)] + (B' - I)[\Phi(t+1)q(t+1) + \Theta(t+1)]. \tag{4.23}$$

$q(t+1)$ can be obtained from equation (4.22),

$$q(t+1) = [I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}Bq(t) -$$

$$[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}\Gamma^{-1}R^{-1}\Gamma'^{-1}\Theta(t+1) -$$

$$[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]\Gamma^{-1}e^d(t). \tag{4.24}$$

Substitute equation (4.24) into (4.23) to obtain

$$\{-\Phi(t) - Q - (B' - I)\Phi(t+1)[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}B\}q(t) =$$

$$\Theta(t) - Qq^d(t) - (B' - I)\Phi(t+1)[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}\Gamma^{-1}R^{-1}\Gamma'^{-1}\Theta(t+1) -$$

$$(B' - I)\Phi(t+1)[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}\Gamma^{-1}e^d(t) + (B' - I)\Theta(t+1). \tag{4.25}$$

From equation (4.25) we have

$$\Phi(t) = (B' - I)\Phi(t+1)[I + \Gamma^{-1}R^{-1}B'^{-1}\Phi(t+1)]^{-1}B + Q, \tag{4.26}$$

which is the matrix Riccati equation for the dynamic Leontief system and

$$\Theta(t) = Qq^d(t) + \{(B' - I)\Phi(t+1)[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}\Gamma^{-1}R^{-1}\Gamma'^{-1} +$$

$$(B' - I)\}\Theta(t+1) + (B' - I)\Phi(t+1)[I + \Gamma^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)]^{-1}\Gamma^{-1}R^{-1}\Gamma'^{-1}e^d(t), \tag{4.27}$$

which is the tracking equation.

Substituting equation (4.21) into equation (4.19) yields

$$e(t) = R^{-1}\Gamma'^{-1}[\Phi(t+1)q(t+1) + \Theta(t+1)] + e^d(t). \tag{4.28}$$

From equations (4.18) and (4.21) we have

$$\lambda(T) = S[q(T) - q^d(T)] = \Phi(T)q(T) + \Theta(T).$$

Therefore

$$\Phi(T) \equiv S, \tag{4.29}$$

$$\Theta(T) \equiv -Sq^d(T), \tag{4.30}$$

which provide the boundary conditions for equations (4.26) and (4.27). Equation (4.28) can be written, using equation (4.6), as

$$e(t) = R^{-1}\Gamma'^{-1}\Phi(t+1)Bq(t) - R^{-1}\Gamma'^{-1}\Phi(t+1)\Gamma^{-1}e(t) +$$

$$R^{-1}\Gamma'^{-1}\Phi(t+1)\Theta(t+1) + e^d(t),$$

or

$$e(t) = [I + R^{-1}\Gamma'^{-1}\Phi(t+1)\Gamma^{-1}]^{-1}R^{-1}\Gamma'^{-1}\Phi(t+1)Bq(t) +$$

$$[I + R^{-1}\Gamma'^{-1}\Phi(t+1)\Gamma^{-1}]^{-1}[R^{-1}\Gamma'^{-1}\Phi(t+1)\Theta(t+1) + e^d(t)]. \tag{4.31}$$

The Riccati equation (4.26) and the tracking equation (4.27), together with the boundary conditions (4.29) and (4.30), determine $\Phi(1)$, $\Phi(2)$, ...,$\Phi(T)$ and $\Theta(1)$, $\Theta(2)$, ..., $\Theta(T)$. Using the matrices $\Phi(1)$ and $\Theta(1)$ together with equation (4.17) we can obtain $e^{opt}(0)$ through equation (4.31). $e^{opt}(0)$, together with $q(0)$, produces $q^{opt}(1)$ using the system dynamic equation (4.16). $\Phi(2)$ and $\Theta(2)$ together with $q^{opt}(1)$ determine $e^{opt}(1)$ through equation (4.31). In general, $\Phi(i)$ and $\Theta(i)$, together with $q^{opt}(i-1)$ which is derived in the $(i-1)$th iteration, determine $e^{opt}(i)$. The iteration carries on until we use $\Phi(T), \Theta(T)$ and $q^{opt}(T-1)$ to find $e^{opt}(T-1)$ which derives the dynamic Leontief model optimally to the final stage of planning.

4.5 Optimal Consumption Policies for the Dynamic Leontief Model using Bellman's Dynamic Programming

Consider again the dynamic Leontief model (4.6), which is restated here as equation (4.32),

$$q(t+1) = Bq(t) - \Gamma^{-1}e(t), \tag{4.32}$$

and the cost function (performance measure) (4.9), restated here as equation (4.33),

$$W = 1/2[q(T) - q^d(T)]'S[q(T) - q^d(T)] +$$
$$1/2 \sum_{t=0}^{T-1} \{[q(t) - q^d(t)]'Q[q(t) - q^d(t)] + [e(t) - e^d(t)]'R[e(t) - e^d(t)]\}, \tag{4.33}$$

where all the terms have been defined earlier. Let us now define

$$q_*(t) = [q_1(t) \;\; q_2(t)...q_n(t) \quad q_n^d(t)...q_1^d(t)]', \tag{4.34}$$

$$e_*(t) = [e_1(t) \;\; e_2(t)...e_m(t) \quad e_m^d(t)...e_1^d(t)]'. \tag{4.35}$$

The cost function, equation (4.33), can, therefore, be written as

$$W = 1/2q_*'(T)S_*q_*(T) + 1/2 \sum_{t=0}^{T-1} [q_*'(t)Q_*q_*(t) + e_*'(t)R_*e_*(t)], \tag{4.36}$$

where

$$S_* = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} & -s_{1n} & \dots & -s_{12} & -s_{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} & -s_{nn} & \dots & -s_{n2} & -s_{n1} \\ -s_{n1} & -s_{n2} & \dots & -s_{nn} & s_{nn} & \dots & s_{n2} & s_{n1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -s_{11} & -s_{12} & \dots & -s_{1n} & s_{1n} & \dots & s_{12} & s_{11} \end{pmatrix},$$

and s_{ij} is the ijth element in the matrix S; and

$$Q_* = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & -a_{1n} & \dots & -a_{12} & -a_{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & -a_{nn} & \dots & -a_{n2} & -a_{n1} \\ -a_{n1} & -a_{n2} & \dots & -a_{nn} & a_{nn} & \dots & a_{n2} & a_{n1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{11} & -a_{12} & \dots & -a_{1n} & a_{1n} & \dots & a_{12} & a_{11} \end{pmatrix},$$

where a_{ij} is the ijth element in the matrix Q. And

$$R_* = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} & -c_{1m} & \dots & -c_{12} & -c_{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} & -c_{nm} & \dots & -c_{n2} & -c_{n1} \\ -c_{n1} & -c_{n2} & \dots & -c_{nm} & c_{nm} & \dots & c_{n2} & c_{n1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_{11} & -c_{12} & \dots & -c_{1m} & c_{1m} & \dots & c_{12} & c_{11} \end{pmatrix},$$

where c_{ij} is the ijth element in the matrix R.[80]

The dynamic Leontief model can be rewritten as the following equation,

$$q_*(t+1) = B_* q_*(t) - \Gamma_*^{-1} e_*(t), \tag{4.37}$$

where

$$B_* = \begin{pmatrix} z_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & z_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & z_n & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & b_{nn} & 0 & \dots & b_{n1} \\ 0 & 0 & 0 & \dots & 0 & b_{(n-1)n} & 0 & \dots & b_{(n-1)1} \\ 0 & 0 & 0 & \dots & 0 & b_{1n} & 0 & \dots & b_{11} \end{pmatrix},$$

and b_{ij} is the ijth element in the matrix B and z_i is the ith diagonal element of matrix Z defined as follows,

$$q^d(t+1) = Zq^d(t).$$

Γ_*^{-1} is given by

$$\Gamma_*^{-1} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_n & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \gamma_{nm} & 0 & \dots & \gamma_{n1} \\ 0 & 0 & 0 & \dots & 0 & \gamma_{(n-1)m} & 0 & \dots & \gamma_{(n-1)1} \\ 0 & 0 & 0 & \dots & 0 & \gamma_{1m} & 0 & \dots & \gamma_{11} \end{pmatrix},$$

where γ_{ij} is the ijth element of the matrix Γ_{-1} and u_i is the ith diagonal element of the matrix U defined as follows,

$$e^d(t+1) = Ue^d(t).$$

The optimal control of the dynamic Leontief model as a problem in servo-mechanism, equations (4.32) and (4.33), is now reduced to a regulator problem, equations (4.36) and (4.37).

[80]Matrices S, Q and R can be assumed to be diagonal.

In what follows, we omit the $*$ from $q_*(t), e_*(t), S_*, Q_*, R_*, B_*$ and Γ_*^{-1} for notational convenience. Define

$$W_0[q(T)] = 1/2q'(T)Sq(T) = W_0^{opt}[q(T)].$$

The cost of operation over the final period is

$$W_1[q(T-1), e(T-1)] =$$

$$1/2q'(T-1)Qq(T-1) + 1/2e'(T-1)Re(T-1) + 1/2q'(T)Sq(T). \qquad (4.38)$$

We have, by definition

$$W_1^{opt}[q(T-1)] = \min_{e(T-1)}\{W_1[q(t-1), e(T-1)]\},$$

which upon using equation (4.37) becomes

$$W_1^{opt}[q(T-1)] = \min_{e(T-1)}\{1/2q'(T-1)Qq(T-1) + 1/2e'(T-1)Re(T-1)+$$

$$1/2[Bq(T-1) + \Gamma^{-1}e(T-1)]'\Phi(0)[Bq(T-1) + \Gamma^{-1}e(T-1)]\}, \qquad (4.39)$$

where $\Phi(0) \equiv S$.

To minimise equation (4.39) with respect to $e(T-1)$ we differentiate the quantities in the brackets and set the derivatives equal to zero. This yields

$$Re(T-1) + \Gamma'^{-1}\Phi(0)[Bq(T-1) + \Gamma^{-1}e(T-1)] = 0. \qquad (4.40)$$

Equation (4.40) may be solved for $e(T-1)$ to give

$$e^{opt}(T-1) = -[R + \Gamma^{-1}\Phi(0)\Gamma^{-1}]^{-1}\Gamma'^{-1}\Phi(0)Bq(t-1),$$

or

$$e^{opt}(T-1) = -\Lambda(T-1)q(T-1), \qquad (4.41)$$

where

$$\Lambda(T-1) = [R + \Gamma'^{-1}\Phi(0)\Gamma^{-1}]^{-1}\Phi(0)B. \qquad (4.42)$$

We now substitute equation (4.41) into equation (4.38) and upon using equation (4.37) we have

$$W_1^{opt}[q(T-1)] = 1/2q'(T-1)Qq(T-1)+$$

$$1/2[\Lambda(T-1)q(T-1)]'R[\Lambda(T-1)q(T-1)]+$$

$$1/2[Bq(T-1)+\Gamma^{-1}\Lambda(T-1)q(T-1)]'\Phi(0)[Bq(T-1)+$$

$$\Gamma^{-1}\Lambda(T-1)q(T-1)] = 1/2q'(T-1)Qq(T-1)+$$

$$1/2[q'(T-1)\Lambda'(T-1)]R[\Lambda(T-1)q(T-1)]+$$

$$1/2q'(T-1)[B+\Gamma^{-1}\Lambda(T-1)]'\Phi(0)[B+\Gamma^{-1}\Lambda(T-1)]q(T-1),$$

or

$$W_1^{opt}[q(T-1)] = 1/2q'(T-1)\{Q+\Lambda'(T-1)R\Lambda(T-1)+$$

$$[B+\Gamma^{-1}\Lambda(T-1)]'\Phi(0)[B+\Gamma^{-1}\Lambda(T-1)]\}q(T-1). \tag{4.43}$$

Equation (4.43) can be written as

$$W_1^{opt}[q(T-1)] = 1/2q'(T-1)\Phi(1)q(T-1), \tag{4.44}$$

where

$$\Phi(1) = Q+\Lambda'(T-1)R\Lambda(T-1)+$$

$$[B+\Gamma^{-1}\Lambda(T-1)]'\Phi(0)[B+\Gamma^{-1}\Lambda(T-1)]. \tag{4.45}$$

The cost function for a two-stage process is

$$W_2[q(T-2),e(T-2),e(T-1)] = 1/2q'(T-2)Qq(T-2)+$$

$$1/2e'(T-2)Re(T-2)+W_1[q(T-1),e(T-1)],$$

which by using the principle of optimality becomes

$$W_2^{opt}[q(T-2)] = \min_{e(T-2)}\{1/2q'(T-2)Qq(T-2)+$$

$$1/2e'(T-2)Re(T-2)+W_1^{opt}[q(T-1)]\}. \tag{4.46}$$

By substituting equation (4.44) into equation (4.46) we obtain

$$W_2^{opt}[q(T-2)] =$$

$$\min_{e(T-2)} \{1/2q'(T-2)Qq(T-2)+1/2e'(T-2)Re(T-2)+1/2q'(T-1)\Phi(1)q(T-1)\},$$

or

$$W_2^{opt}[q(T-2)] =$$

$$\min_{e(T-2)} \{1/2q'(T-2)Qq(T-2) + 1/2e'(T-2)Re(T-2) + 1/2[q'(T-2)B'+$$

$$e'(T-2)\Gamma'^{-1}]\Phi(1)[Bq(T-2) + \Gamma^{-1}e(T-2)\}. \tag{4.47}$$

Taking derivatives with respect to $e(T-2)$ yields

$$Re(T-2) + \Gamma'^{-1}\Phi(1)[Bq(T-2) + \Gamma e(T-2)] = 0, \tag{4.48}$$

which is similar to equation (4.40). From equation (4.48) we can derive the optimal control at stage two as follows,

$$e^{opt}(T-2) = -[R + \Gamma'^{-1}\Phi(1)\Gamma^{-1}]^{-1}\Gamma'^{-1}\Phi(1)Bq(T-2),$$

which is similar to equation (4.41). Alternatively, we have

$$e^{opt}(T-2) = -\Lambda(T-2)q(T-2), \tag{4.49}$$

where

$$\Lambda(T-2) = [R + \Gamma'^{-1}\Phi(1)\Gamma^{-1}]^{-1}\Gamma'^{-1}\Phi(1)B.$$

If we now substitute equation (4.49) back into equation (4.46), we will obtain

$$W_2^{opt}[q(T-2)] =$$

$$1/2q'(T-2)\{[B + \Gamma^{-1}\Lambda^{-1}(T-2)]'\Phi(1)[B + \Gamma^{-1}\Lambda(T-2)]+$$

$$\Lambda'(T-2)R\Lambda(T-2) + Q\}q(T-2), \tag{4.50}$$

which can be written as

$$W_2^{opt}[q(T-2)] = 1/2q'(T-2)\Phi(2)q(T-2), \tag{4.51}$$

where

$$\Phi(2) = [B + \Gamma^{-1}\Lambda(T-2)]'\Phi(1)[B + \Gamma^{-1}\Lambda(T-2)]+$$

$$\Lambda'(T-2)R\Lambda(T-2) + Q. \tag{4.52}$$

By induction, we obtain the following relation,

$$e^{opt}(t) = -[R + \Gamma'^{-1}\Phi(t-1)\Gamma^{-1}]^{-1}\Gamma'^{-1}\Phi(t-1)Bq(t), \tag{4.53}$$

or

$$e^{opt}(t) = -\Lambda(t)q(t), \qquad (4.54),$$

where

$$\Lambda(t) = [R + \Gamma'^{-1}\Phi(t-1)\Gamma^{-1}]^{-1}\Gamma'^{-1}\Phi(t-1)B, \tag{4.55}$$

and

$$W^{opt}_{N-k}[q(t)] = 1/2q'(t)\{Q + \Lambda'(t)R\Lambda(t)+$$

$$[B + \Gamma^{-1}\Lambda(t)]'\Phi(t-1)[B + \Gamma^{-1}\Lambda(t)]\}q(t), \tag{4.56}$$

or

$$W^{opt}_{N-k}[q(t)] = 1/2q'(t)\Phi(t)q(t), \tag{4.57}$$

where

$$\Phi(t) = Q + \Lambda'(t)R\Lambda(t) + [B + \Gamma^{-1}\Lambda(t)]'\Phi(t-1)[B + \Gamma^{-1}\Lambda(t)]. \tag{4.58}$$

Equation (4.58) is the discrete matrix version of the *Riccati* equation for the dynamic Leontief model.

To summarise, the deterministic optimal control of the dynamic Leontief model can be obtained by the following steps:

i) Evaluate $\Lambda(T-1)$ from equation (4.55) using $\Phi(0) \equiv S$.

ii) Substitute $\Lambda(T-1)$ into the Riccati equation (4.58) in order to compute $\Phi(1)$. Evaluate $\Lambda(T-2)$ from equation (4.55) using $\Phi(1)$ and substitute it into equation (4.58) to obtain $\Phi(2)$. Continue this process until all values of $\Lambda(T-1)$, $\Lambda(T-2)$, ..., $\Lambda(0)$ and $\Phi(0)$, $\Phi(1)$, ..., $\Phi(T-1)$ are computed.

iii) The optimal control for the base period $e^{opt}(0)$ can be evaluated by using equation (4.54) together with the value of $q(0)$. $e^{opt}(0)$ can be used to compute $q^{opt}(1)$ through equation (4.37).

iv) Substitute $q^{opt}(1)$ and $\Lambda(1)$ into equation (4.54) to determine $e^{opt}(1)$. Continue this process until the optimal sequence of final demand $e^{opt}(0)$, $e^{opt}(1)$, ... and $e^{opt}(T-1)$ and the optimal trajectory of output $q^{opt}(1)$, $q^{opt}(2)$, ... and $q^{opt}(T)$ have been determined.

4.6 Optimal Control of the Dynamic Leontief Model with Substitution: Problem Formulation

Dynamic input-output analysis using Leontief models with substitution has been well-known for many years. Dantzig (1955), Koopmans (1951), and Samuelson (1951) were the first writers to use a linear programming approach to the Leontief model with substitution. In the remainder of this chapter we assume a substitution system in which there exists an uncertainty about the system's future state resulting from the current decisions on substitute activities. For greater generality, we assume the presence of measurement errors in the substitution system. The state of the substitution system is, therefore, described by a conditional probability density which should be updated, at every stage, according to *Bayes' rule.* The generalized stochastic dynamic programming functional recurrence equation, developed by Jacobs (1977), has been used to obtain the optimal substitute activities.

The optimal solution for the substitution system, as described above, is regarded as an input to the greater problem of stochastic control of a Leontief dynamic system. The stochastic dynamic Leontief model is assumed to have a disturbance term affecting its future state. A quadratic cost function is used and the dynamic stochastic Leontief model with substitution is then solved as a tracking problem in stochastic control theory.

Let us consider a discrete time Leontief dynamic system in which we observe the behaviour of the model at discrete points in time, t=0, 1, 2, ..., T. At time t we are given an n-dimensional vector $g(t)$ where $g(t) \in R^n$ and $t \in \{1, 2, ..., T\}$. $g(t)$ can be interpreted as the given bill of final goods to be produced. The set of possible decisions at time t is given by $d(t)$ where

$$d(t)|\{A(d_t)q(t) = g(t)\}. \tag{4.59}$$

The system described by equation (4.59) is the *Leontief Substitution System.* In what follows we derive a policy sequence which generates optimal substitute activities such that a cost function, which will be defined later, is minimised. In other words, the solution to the system need not be unique because the set of different policies can determine alternative substitute activities in the Leontief model. The essential problem is to choose the values of activity levels such that a *cost function* is minimised. Arrow (1951), Samuelson (1951) and Morishima (1965) have solved, for the first time, the Leontief substitution model with a *linear cost function.*

In this chapter, we will derive the optimal policies for a Leontief substitution system with *non-linear dynamics* and subject to a *non-linear cost function.* Moreover, we assume a *stochastic* non-linear Leontief substitution system in which there exists uncertainties about the state resulting from a given policy. In order to achieve a greater generality, we solve the optimal non-linear stochastic Leontief substitution system with measurement errors as well as future uncertainties.

Suppose that at time t there are n commodities to be produced, $q_1(t)$, $q_2(t)$,

..., $q_n(t)$, with the corresponding current prices $p_1(t)$, $p_2(t)$, ..., $p_n(t)$. It is desired to produce the total output vector $q(t)$ such that a given bill of final goods is satisfied. We assume that there are k different substitute activities, therefore

$$A(d_t) = \{A(d_t^1), A(d_t^2), ..., A(d_t^k)\}.$$

If the policy $A(d_t^1)$ is chosen, the total cost of intermediate goods necessary to produce $q(t)$, which is sufficient to generate $g(t)$, will be

$$c^1(t) = p'(t)[I - A(d_t^1)]q^1(t), \tag{4.61}$$

where $p(t)$ is the n-dimensional price vector and p' denotes the transpose of p. Applications of policies $A(d_t^2)$, $A(d_t^3)$, ..., $A(d_t^k)$ will result in $c^2(t)$, $c^3(t)$, ..., $c^k(t)$ through equation (4.61). Among the set of possible policies, $A(d_t^i)$, i=1, 2, ..., k, the optimal policy is that which generates the minimum cost.

We now assume that the policy $A(d_t)$, taken in a particular period (a year), can be used as an argument in a function for the price vector $p(t+1)$ the following year. Such an assumption can be justified on the ground that the policy taken in any year influences the demand for different commodities in that year and the supply pattern of goods in the next year. More specifically, the optimal policy next year will depend on the price structure next year and the latter is a function of the current policy. The policy next year will not, therefore, be independent of what we are going to do this year. It follows that an optimising solution is necessary over the entire period of planning substitute activities. Note that the price vector next year, $p(t + 1)$, is a function of a number of variables including $A(d_t)$. With no loss of generality, we concentrate on the following simple function,

$$p(t + 1) = h[p_t, A(d_t)].$$

Moreover, we assume the presence of uncertainties in future value of $p(t)$ when a policy $A(d_t)$ is chosen. The dynamics of a stochastic non-linear Leontief substitution system is, therefore, described by

$$p(t+1) = h[p_t, A(d_t), u_1(t)], \tag{4.62}$$

where $u_1(t)$ is an n-dimensional vector of random disturbances at time t with a given probability distribution. The price vector at time zero, $p(0)$, is assumed to be fully specified.

It is also assumed that there exists an uncertainty about the present value of the price vector $p(t)$, that is, the price vector cannot be measured exactly and the available measurements are, therefore, noisy. The dynamics of the measurement subsystem in a Leontief substitution system can be formulated as follows,

$$p^*(t) = h^*[p(t), u_2(t)], \tag{4.63}$$

where $u_2(t)$, a j-dimensional random vector with given probability distribution, represents the disturbances in the measurement sub-system. The functions h and h^* are assumed to be completely specified over the entire planning horizon and $u_1(t)$ and $u_2(t)$ are independent sequence with zero means and are independent, jointly Gaussian random vectors.

For the class of stochastic substitution systems with uncertainty about the future state, equation (4.62) and with measurement error, equation (4.63), the following cost function is assumed,

$$W_N = \sum_{t=0}^{T} \phi[p(t), A(d_t)], \tag{4.64}$$

where W_N is the cost of intermediate products over N-stages of planning. Any decision chosen at time zero, $A(d_0)$, will, together with the specified $p(0)$, generate the cost, $W(0)$, according to equation (4.64). $A(d_0)$ will affect the price vector $p(1)$ next year using equation (4.62) and any policy chosen next year, with regard to the price vector at that time, will generate the cost $W(1)$, at time one. This procedure continues until the policy $A(d_{T-1})$ is chosen which generates the price vector $p(T)$ for the last stage of the planning. The objective is to find a policy sequence which minimises the total cost over the entire planning period.

Since the value of ϕ in equation (4.64) depends on *i*) the uncertain current value of price vector $p(t)$ and *ii*) the uncertain future values of the random disturbance vectors $u_1(t)$ and $u_2(t)$, the summation of ϕ over t=0, 1, 2, ..., T will be a random scalar. In order to provide a deterministic cost function we need to take the expected value V of W with respect to the probability distributions of all stochastic variables,

$$V_N(t) = E_{p(t),u^t}[W(t)], \tag{4.65}$$

where u^t is the future sequence of the disturbance vectors, $u_1(t)$ and $u_2(t)$, defined as follows,

$$u^t \equiv \{u_1(t), u_1(t+1), ..., u_1(T-1), u_2(t+1), ..., u_2(T-1)\}.$$

$u_1(T)$ and $u_2(T)$ are not included in the sequence because they do not affect the total cost defined by equation (4.64). The sequence for u_2 starts at $u_2(t+1)$ since we have already obtained the conditional probability density for the current price vector $p(t)$. The optimization problem for a Leontief stochastic non-linear substitution system with both future uncertainty and measurement error is to find a policy sequence $\{A[d(t)], t = 0, 1, ..., T-1\}$ which minimises the expected value of the cost function defined by equation (4.65).

4.7 A Dynamic Programming Solution of the Leontief Substitution System

The current price vector in a Leontief substitution system is described by a probability density function. This is because measurement errors are present in the price vector $p(t)$. Such a probability distribution for the current price vector will be conditional on the past measurements of price vector $p^*(i)$, i=1, 2, ..., t and on the past policies chosen on substitute activities, $A(d_i)$, i=0, 1, ..., t-1. If we denote such a history of information, available at time t, by α^t then

$$\alpha^t = \{A(d_0), A(d_1), ..., A(d_{t-1}), p^*(1), p^*(2), ..., p^*(t)\}.$$

In α^t the policy sequence terminates at time $t-1$ because the policy chosen at time t has no effect on current price vector and the measurement sequence starts at time $t = 1$ since it is assumed that the initial measurement $p^*(0)$ is already used to construct prior probability for the initial price vector $p(0)$. Using the notation α^t, the conditional probability distribution for the current price vector can be summarised as

$$\Pr[p(t)|\alpha^t].$$

For a Leontief substitution system described by equation (4.62), the conditional probability density function for the price vector can be updated at each stage by *Bayes' rule.* The optimal policy $A^{opt}(d_t)$ will be chosen with regard to the updated conditional probability density for the current price vector $p(t)$.

The conditional probability density for the price vector can be updated at each stage according to Bayes' rule, i.e.

$$\Pr[p(t)|\alpha^t] = \frac{\Pr[p(t)]\Pr[p^*(t)|p(t)]}{\int_{p(t)} \Pr[p(t)]\Pr[p^*(t)|p(t)]dp(t)}, \tag{4.66}$$

where $\Pr[(p(t)]$ is the *prior probability* density for the random price vector $p(t)$ before the noisy measurement $p^*(t)$ is made. $\Pr[p^*(t)|p(t)]$ is the *likelihood function* and specifies the relative probabilities of the noisy price measurement vector $p^*(t)$, taken its observed value as a function of the range of possible values $p(t)$. $\Pr[p(t)|p^*(t)]$ is the *conditional* probability density for the random price vector $p(t)$ after the noisy measurement $p^*(t)$ is made.

If $V_N^{opt}\{\Pr[p(t)|p^*(t)], T\}$ is the expected value of W, as defined by equation

(4.65) and obtained by using optimal policies over N-stages, starting from the uncertain state at time $(T - N + t)$ described by the conditional probability for price vector $\Pr[p(T - N + t)|p^*(t)]$ and terminating at time T, then

$$V_N^{opt}(t) \equiv \min_{A(d)^t}[V_N(t)], \tag{4.67}$$

where $V_N^{opt}(t)$ is a shorthand for $V_N^{opt}\{\Pr[p(t)|p^*(t)], T\}$ and $A(d)^t$ is the policy sequence defined as

$$A(d)^t = \{A[d(t)], ..., A[d(t + N - 1)]\}.$$

$V_N^{opt}(t)$ in now a functional rather that a function. It includes as one of its argument the probability density function for the price vector $\Pr[p(t)|p^*(t)]$, which is itself a function of the uncertain price vector. The dynamic programming recurrence equation hence can be written in terms of such a functional. Applications of Bellman's principle of optimality yields the following recurrence equation,

$$V_N^{opt}(t) = \min_{A(d_t)}\{E_{p(t),u_1(t)}[\phi(t + 1) + V_{N-1}^{opt}(t + 1)]\}. \tag{4.68}$$

Equation (4.68) determines the manner in which optimal policies $A(d)^t$ for the Leontief substitution system can be produced.

For linear systems with a quadratic cost function and Gaussian random variable, there exists an analytic solution to the functional recurrence equation [see, for example, Dreyfus (1965), Dreyfus and Kan (1973)]. For the class of non-linear systems with a general non-linear cost function such as the Leontief substitution system presented here, an appropriate search technique can be employed to derive the optimal policies using equation (4.68).

In what follows, we derive the functional recurrence equation for the stochastic non-linear Leontief substitution system. We employ Åstrom's *fundamental lemma of stochastic control theory* [Åstrom (1970)] as generalized by Jacobs (1977) for non-linear systems. The total cost of intermediate product generated by policy sequence $A(d)^t$ can be divided into two components: the cost associated with stage $(t+1)$, namely $\phi(t+1)$ and the subsequent cost, $W_{N-1}(t+1)$, where W_{N-1} indicates that the cost is summed over the $N-1$ remaining stages, i.e.

$$W_N(t) = \phi(t+1) + W_{N-1}(t+1). \tag{4.69}$$

Equation (4.69), by using equations (4.65) and (4.67), can be written as

$$V_N^{opt}(t) = \min_{A(d)^t}\{E_{p(t),u^t}[\phi(t+1) + W_{N-1}(t+1)]\}, \tag{4.70}$$

or

$$V_N^{opt}(t) = \min_{A(d)^t}\{E_{p(t),u^t}[\phi(t+1)] + E_{p(t),u^t}[W_{N-1}(t+1)]\}. \tag{4.71}$$

The total cost of intermediate goods $\phi(t+1)$, at time $t+1$, depends only on the policy chosen at time t and, therefore, is independent of the decisions taken at time $t+i$, $i>0$. The cost at time $t+1$ is also independent of the future sequences of random vectors u_1 and u_2. Note that the disturbance term $u_2(t+1)$ in the price measurement subsystem at time $t+1$ does not affect the cost of intermediate goods at time $t+1$. Hence, we have

$$E_{p(t),u^t}[\phi(t+1)] = E_{p(t),u(t)}[\phi(t+1)]. \tag{4.72}$$

Using equation (4.72), equation (4.71) can be written as follows,

$$V_N^{opt}(t) = \min_{A(d_t)}\{E_{p(t),u_1(t)}[\phi(t+1)] + \min_{A(d)^{t+1}}\{E_{p(t),u^t}[W_{N-1}(t+1)]\}\}. \tag{4.73}$$

Using Åstrom's fundamental lemma of stochastic control theory, we will show that

$$\min_{A(d)^{t+1}}\{E_{p(t),u^t}[W_{N-1}(t+1)]\} = V_{N-1}^{opt}(t+1), \tag{4.74}$$

where $A(d)^{t+1}$ is the policy sequence starting from time $t+1$ and terminating at the end of the planning period.

The two following lemmas are stated without proof.[81] Consider the functional minimization,

$$\min_r\{E_k[f(k,r)]\}, \tag{4.75}$$

where k is a random state variable with a given probability density $\Pr(k)$ and r is a control variable which is to be chosen to minimise the mean value of f. Such a difficult functional minimization can be simplified by conversion to a series of ordinary minimization.

Lemma 1 For the class of functional minimization expressed in equation (4.75), the operation of minimization and expectation commute. Thus we have

$$\min_r\{E_k[f(k,r)]\} = E_k\{\min_r[f(k,r)]\}. \tag{4.76}$$

The following lemma is the generalization of Lemma 1 when the scalar function f is a function of more than one random variable. Consider the case of two random variables k_1 and k_2 with given joint probability density $\Pr(k_1, k_2)$ and r the control variable which may be chosen to minimise the mean value of f.

Lemma 2 The minimum of the joint mean is the same as marginal mean of the minimum of each conditional mean,

$$\min_r[E_{k_1,k_2}(f)] = E_{k_2}\{\min_r[E_{k_1}(f|k_2)]\} = E_{k_1}\{\min_r[E_{k_2}(f|k_1)]\}, \tag{4.77}$$

where the joint mean is defined as

$$E_{k_1,k_2}(f) = \int_{k_1}\int_{k_2} f_1(k_1,k_2,r)\Pr(k_1,k_2)dk_1dk_2.$$

In what follows, these two lemmas are applied to the optimal cost function for the Leontief substitution system. The expectation in equation (4.74) is over the random price vector $p(t)$, the sequence of disturbances in the substitution system

[81]The proof is given, for example, in Åstrom (1970).

(4.62) and the measurement system (4.63). If we now partition $p(t)$ and u^t into the following two groups

$$k_1 \equiv p(t), u_1(t), \qquad and \qquad k_2 \equiv u_2(t+1), u^{t+1},$$

then according to equation (4.77) we will have

$$\min_{A(d)^{t+1}} \left\{ E_{p(t),u^t}[W_{N-1}(t+1)] \right\} =$$

$$E_{p(t),u_1(t)} \left[\min_{A(d)^{t+1}} \left\{ E_{u_2(t+1),u^{t+1}}[W_{N-1}(t+1)|p(t), u_1(t)] \right\} \right]. \tag{4.78}$$

$p(t)$ and $u_1(t)$ are conditioning variables in equation (4.78) and as time passes and they take their specific values, $u_2(t+1)$ will then determine the next updated probability distribution for the price vector $p(t+1)$. Hence

$$E_{u_2(t+1),u^{t+1}} \left\{ W_{N-1}(t+1)|p(t), u_1(t) \right\} =$$

$$E_{p(t+1),u^{t+1}} \left\{ W_{N-1}(t+1)|p(t), u_1(t) \right\}, \tag{4.79}$$

and its minimum with respect to policies $A(d)^{t+1}$ is

$$\min_{A(d)^{t+1}} \left\{ E_{p(t+1),u^{t+1}}[W_{N-1}(t+1)|p(t), u_1(t)] \right\} \equiv$$

$$V_{N-1}^{opt}[(t+1)|p(t), u_1(t)]. \tag{4.80}$$

Substituting equations (4.80) and (4.78) into equation (4.73) gives

$$V_N^{opt}(t) = \min_{A(d_t)} \{ E_{p(t),u_1(t)}[\phi(t+1)] + E_{p(t),u_1(t)} \{ V_{N-1}^{opt}[(t+1)|p(t), u_1(t)] \} \}. \tag{4.81}$$

By rearranging the above equation, we have

$$V_N^{opt}(t) = \min_{A(d_t)} \{ E_{p(t),u_1(t)}[\phi(t+1) + V_{N-1}^{opt}(t+1)] \}, \tag{4.82}$$

where the dependency of $V_{N-1}^{opt}(t+1)$ on the values of $p(t)$ and $u_1(t)$ is not explicitly shown.

Equation (4.82) is the functional recurrence equation for the stochastic non-linear Leontief substitution system with measurement error and uncertainty about future states of the system . It formulates the optimal policy, at any time t, to achieve the minimum value of the total cost of intermediate products over the entire planning horizon.

4.8 Optimal Stochastic Control of Dynamic Leontief Models with Future Uncertainties and a Stochastic Substitution

A dynamic Leontief model with uncertainty about its future state is considered as a tracking problem in optimal stochastic control theory. A quadratic objective function is used in which any deviation of the total output vector from its desired trajectory is to be minimised. It is assumed that there is no measurement error in the system. A stochastic substitution subsystem with non-linear dynamics and uncertainty about both the present and future states is incorporated in the stochastic dynamic Leontief system. The optimal sequence of policies for the substitution system, developed in section 4.7, is regarded as an input to the controller of the stochastic dynamic Leontief model which updates the Leontief matrix at each planning period.

The stochastic dynamic Leontief model is described by the following difference equation,

$$q(t+1) = B(t)q(t) - \Gamma^{-1}(t)e(t) + \xi(t), \tag{4.83}$$

where $q(t) \in R^n$, $e(t) \in R^m$ and R^n denotes the n-dimensional Euclidean space. We also have t=0, 1, ... , T. $\xi(t)$ is an n-vector of state disturbances and all other term are defined earlier. $\xi(t)$ is assumed to be a white random disturbance vector with zero mean and covariance matrix $\Xi(t)$. The problem is to find a control sequence

$$e^{opt}(t) = \{e^{opt}(0), e^{opt}(1), ..., e^{opt}(T-1)\}$$

which minimises the following cost function,

$$\begin{aligned} W_N = {} & 1/2[q(T) - q^d(T)]'S[q(T) - q^d(T)] + \\ & 1/2\sum_{t=0}^{T-1}\{[q(t) - q^d(t)]'Q(t)[q(t) - q^d(t)] + \\ & [e(t) - e^d(t)]'R(t)[e(t) - e^d(t)]\}, \end{aligned} \tag{4.84}$$

where q^d and e^d are desired values of vectors q and e, respectively.

The stochastic dynamic substitution subsystem defined in equations (4.62-4.64) is included in the Leontief dynamic model, equation (4.83). The optimal sequence of decisions $[d_0, d_1, ..., d_{T-1}]$ obtained from equation (4.82) will generate optimal base matrices $A(0), A(1), ..., A(T)$, which will enter equation (4.83).

The optimal stochastic dynamic Leontief model with substitution, presented here as a tracking problem, can be easily translated into a regulator problem by defining new and higher order matrices for B, Γ, S, R and Q. The mathematics of such a transformation is given in section 4.5 and is not repeated here. For notational and algebraic convenience, but with no loss of generality, the Leontief model will be discussed as a regulator problem.

4.8.1 A Dynamic Programming Formulation of the Problem

The information available at time t for deriving the optimal control solution for a stochastic Leontief system consists of the output history, q^t and the past final consumption sequence, e^{t-1}, defined as

$$q^t = \{q(0), q(1), ..., q(t)\},$$

$$e^{t-1} = \{e(0), e(1), ..., e(t-1)\}.$$

Since the value of W in equation (4.84) depends on the uncertain future values of the random disturbance vector $\xi(t)$, its value will be a random scalar. In order to provide a deterministic cost function one needs to take the expected value of W with respect to the probabilistic variables. Let us define $V_{N-k}^{opt}(t)$ as the optimal cost over $(N-k)$ periods from time t to the end of optimization process at time T. We, therefore, have

$$V_{N-k}^{opt}(t) = \min_{e(t),e(t+1),\ldots,e(T-1)} \{E_{q(t),\ldots,q(T)}\{\sum_{t=t}^{T-1}[q'(t)Q(t)q(t)+$$

$$e'(t)R(t)e(t)] + q'(T)S(T)q(T)|q^t\}\}. \tag{4.85}$$

Equation (4.85) can be written as

$$V_{N-k}^{opt}(t) = \min_{e(t),e(t+1),\ldots,e(T-1)} \{E_{q(t),\ldots,q(T)}\{q'(t)Q(t)q(t)+$$

$$e'(t)R(t)e(t) + \sum_{t=t+1}^{T-1}[q'(t)Q(t)q(t) + e'(t)R(t)e(t)]+$$

$$q'(T)S(T)q(T)|q^t\}\}. \tag{4.86}$$

The two terms $q'(t)Q(t)q(t)$ and $e'(t)R(t)e(t)$ can be taken out of brackets because $e(t+1)$, ..., $e(T-1)$ will not affect them. Therefore

$$V_{N-k}^{opt}(t) = \min_{e(t)}\{[q'(t)Q(t)q(t) + e'(t)R(t)e(t)]+$$

$$\min_{e(t+1),\ldots,e(T-1)}\{E_{q(t+1),\ldots,q(T)}\{\sum_{t=t+1}^{T-1}[q'(t)Q(t)q(t)+$$

$$e'(t)R(t)e(t)] + q'(T)S(T)q(T)|q^t, e(t)\}\}\}. \tag{4.87}$$

By using the property of conditional expectation, namely,

$$E_{r,k}[f(r,k)] = E_r\{E_k[f(r,k)|r]\},$$

equation (4.87) can be written as follows,

$$V_{N-k}^{opt}(t) = \min_{e(t)}\{q'(t)Q(t)q(t) + e'(t)R(t)e(t)+$$

$$\min_{e(t+1),\dots,e(T-1)}\{E_{q(t+1)}\{E_{q(t+2),\dots,q(T)}\{\sum_{t=t+1}^{T-1}[q'(t)Q(t)q(t)+$$

$$e'(t)R(t)e(t)]+q'(T)S(T)q(T)|q^t,q(t+1),e(t)\}|q^t\}\}\}. \tag{4.88}$$

Using Åstrom's fundamental lemma of stochastic control theory, equation (4.76), we rewrite equation (4.88) as follows,

$$V^{opt}_{N-k}(t)=\min_{e(t)}\{q'(t)Q(t)q(t)+e'(t)R(t)e(t)+$$

$$E_{q(t+1)}\{\min_{e(t+1),\dots,e(T-1)}\{E_{q(t+2),\dots,q(T)}\{\sum_{t=t+1}^{T-1}[q'(t)Q(t)q(t)+$$

$$e'(t)R(t)e(t)]+q'(T)S(T)q(T)]|q^t,q(t+1),e(t)\}\}\}\}. \tag{4.89}$$

We can now write an equation for $V^{opt}_{N-k-1}(t+1)$ analogous to equation (4.85) and thus rewrite equation (4.89) as follows,

$$V^{opt}_{N-k}(t)=\min_{e(t)}\{q'(t)Q(t)q(t)+e'(t)R(t)e(t)+E_{q(t+1)}[V^{opt}_{N-k-1}(t+1)]\}. \tag{4.90}$$

Equation (4.90) is the fundamental dynamic programming recurrence equation for the dynamic Leontief model incorporating uncertainties only about its future. Equation (4.90) must be solved backwards beginning with the following terminal condition,

$$V^{opt}(T)=E\{q'(T)S(T)q(T)|q^T,e^{T-1}\}.$$

Note that the stochastic dynamic Leontief model, equation (4.83), implies that the output history, q^t, at time t provides no information about $q(t+1)$ which is not contained in $q(t)$. Hence

$$E_{q(t)}[V^{opt}_{N-k-1}(t+1)|q^t,e(t)]=E_{q(t)}[V^{opt}_{N-k-1}(t+1)|q(t),e(t)]. \tag{4.91}$$

Assuming that the optimal cost for dynamic Leontief models with linear dynamics and quadratic performance criterion is quadratic, we have

$$V^{opt}_{N-k}(t)=q'(t)\Phi(t)q(t)+\epsilon(t), \tag{4.92}$$

where $\epsilon(t)$ is assumed to be independent of $q(t)$ and $\Phi(t)$ is a symmetric n-dimensional matrix. We now substitute equation (4.92) into the functional recurrence equation (4.90) to obtain the following equation,

$$V_{N-k}^{opt}(t) = \min_{e(t)}\{q'(t)Q(t)q(t) + e'(t)R(t)e(t) +$$

$$E_{q(t+1)}[q'(t+1)\Phi(t+1)q(t+1) + \epsilon(t+1)|q(t), e(t-1)]\}. \qquad (4.93)$$

In order to write the following term

$$E_{q(t+1)}[q'(t+1)\Phi(t+1)q(t+1)] \qquad (4.94)$$

in terms of the mean m and covariance matrix σ of the random variable q, we first rewrite expression (4.94) as follows,

$$E[q'(t+1)\Phi(t+1)q(t+1)] =$$

$$E\{[q(t+1) - m(t+1)]'\Phi(t+1)[q(t+1) - m(t+1)] +$$

$$m'(t+1)\Phi(t+1)q(t+1) + q'(t+1)\Phi(t+1)m(t+1) -$$

$$m'(t+1)\Phi(t+1)m(t+1)\}. \qquad (4.95)$$

Since $E(q) = m$, equation (4.95) can be written as

$$E[q'(t+1)\Phi(t+1)q(t+1)] =$$

$$E[q(t+1) - m(t+1)]'\Phi(t+1)[q(t+1) - m(t+1)] + m'(t+1)\Phi(t+1)m(t+1). \qquad (4.96)$$

Since

$$E\{[q(t+1) - m(t+1)]'\Phi(t+1)[q(t+1) - m(t+1)]\} =$$

$$tr\{\Phi(t+1)[q(t+1) - m(t+1)][q(t+1) - m(t+1)]'\}, \qquad (4.97)$$

substituting equation (4.97) into equation (4.96) yields

$$E[q'(t+1)\Phi(t+1)q(t+1)] = tr[\Phi(t+1)\sigma(t+1)] +$$

$$m'(t+1)\Phi(t+1)m(t+1), \qquad (4.98)$$

which makes equation (4.93) as follows,

$$V^{opt}_{N-k}(t) = \min_{e(t)}\{q'(t)Q(t)q(t) + e'(t)R(t)e(t)+$$

$$tr\{\Phi(t+1)\sigma(t+1)] + m'(t+1)\Phi(t+1)m(t+1) + \epsilon(t+1)\}. \tag{4.99}$$

Using the dynamic stochastic Leontief model, together with the knowledge that $E[q(t+1)] = m(t+1)$ and $E[\epsilon(t)] = 0$, equation (4.99) can be written as

$$V^{opt}_{N-k}(t) = \min_{e(t)}\{q'(t)Q(t)q(t) + e'(t)R(t)e(t)+$$

$$[B(t)q(t) - \Gamma^{-1}(t)e(t)]'\Phi(t+1)[B(t)q(t) - \Gamma^{-1}(t)e(t)]+$$

$$tr[\Phi(t+1)\sigma(t+1)] + \epsilon(t+1)\}. \tag{4.100}$$

By completing the square for the terms containing $e(t)$, equation (4.100) becomes

$$V^{opt}_{N-k}(t) = \min_{e(t)}\{[e(t) + \Lambda(t)q(t)]'[\Gamma'^{-1}(t)\Phi(t+1)\Gamma^{-1}+$$

$$R(t)][e(t) + \Lambda(t)q(t)] + q'(t)\{Q(t) + B'(t)\Phi(t+1)B(t)-$$

$$\Lambda'(t)[\Gamma'^{-1}(t)\Phi(t+1)\Gamma^{-1}(t) + R(t)\Lambda(t)\}q(t)+$$

$$tr[\Phi(t+1)\sigma(t+1)] + \epsilon(t+1)\}, \tag{4.101}$$

where

$$\Lambda(t) = [\Gamma'^{-1}(t)\Phi(t+1)\Gamma^{-1}(t) + R(t)]^{-1}\Gamma'(t)\Phi(t+1)B(t), \tag{4.102}$$

is the *gain matrix* for the stochastic dynamic Leontief system. If we examine equation (4.101) it will become obvious that the minimum cost can be achieved if

$$e^{opt}(t) = -\Lambda(t)q(t). \tag{4.103}$$

Let us now substitute equations (4.92) and (4.102) and the minimising value of final consumption $e^{opt}(t)$ equation (4.103), into equation (4.101), to obtain

$$q'(t)\Phi(t)q(t) + \epsilon(t) = q'(t)\{Q(t) + B'(t)\Phi(t+1)B(t)-$$

$$B'(t)\Phi'(t+1)\Gamma^{-1}(t)[\Gamma'^{-1}(t)\Phi(t+1)\Gamma^{-1}(t)+R(t)]^{-1}\times$$

$$\Gamma'^{-1}(t)\Phi(t+1)B(t)\}q(t)+tr[\Phi(t+1)\sigma(t+1)]+\epsilon(t+1). \tag{4.104}$$

It follows from equation (4.104) that

$$\Phi(t)=Q(t)+B'(t)\Phi(t+1)B(t)-$$

$$B'(t)\Phi'(t+1)\Gamma^{-1}(t)[\Gamma'^{-1}(t)\Phi(t+1)\Gamma^{-1}(t)+R(t)]^{-1}\Gamma'^{-1}(t)\Phi(t+1)B(t), \tag{4.105}$$

which is the matrix Riccati equation for the stochastic dynamic Leontief model with exact measurement and

$$\epsilon(t)=tr[\Phi(t+1)\sigma(t+1)]+\epsilon(t+1). \tag{4.106}$$

The boundary conditions for the solution of equations (4.105) and (4.106) are as follows,

$$\Phi(T)=S, \tag{4.107}$$

and

$$\epsilon(T)=0. \tag{4.108}$$

Equations (4.102) and (4.103), together with equations (4.105)-(4.108), completely determine the optimal stochastic control solution for a dynamic Leontief model with future uncertainty and quadratic objective function. The stochastic substitution system which is incorporated in the dynamic Leontief model operates independently of the Leontief controller in obtaining the optimal policies. The functional recurrence equation for the stochastic substitution system, equation (4.82), generates the sequence of optimal base matrices and feeds them to the process formulated to find the optimal stochastic control of the dynamic Leontief model. The immediate effect of this operation is that the matrix A and all the matrices depending on it, such as B, become time-variant.

Equations (4.102) and (4.103), which give the optimal sequence of final consumption for the stochastic Leontief model with future uncertainty, are exactly the same as those equations which give the same control solution for a deterministic dynamic Leontief model. Addition of the white random forcing function, $\xi(t)$, which can be interpreted as the future output uncertainty associated with any sequence of final consumption does not change the optimal control policy as obtained in the deterministic case. It adds, however, a scalar term to the minimum expected cost.

Although the analytical optimal control solutions are the same, there exists an important difference between optimal control of deterministic dynamic Leontief model and that of the stochastic model with future uncertainty. In the deterministic case, the dynamics of the Leontief model can usually be expressed as

$$q(t+1) = B(t)q(t) + \Gamma^{-1}(t)e(t),$$

and, therefore, if we know the initial output vector, $q(t=0)$ and the sequence of final consumption $e(t)$, t=0, 1, 2, ..., T-1, the state of the model will become completely specified. Hence, a control law can be formulated as a function of the initial output vector $q(0)$ and time t, i.e. $e^{opt}[q(0), t]$ and thus need not be a feedback control. This is not, however, true for the stochastic control of a dynamic Leontief model since there exists a random disturbance vector $\xi(t)$ in the system dynamics. This explains why in the stochastic case, the optimal sequence of final consumption must be defined as a function of output and time, i.e. $e^{opt}[q(t), t]$. Feedback control is, therefore, an essential condition in stochastic control of Leontief models.

4.9 Summary and Concluding Remarks

In this chapter, we have, for the first time, formulated and solved dynamic deterministic and stochastic Leontief models with and without a substitution system. Section 4.3 is concerned with formulating input-output models as a control process in which the system dynamic equations for the Leontief dynamic model is expressed as a tracking (or servo-mechanism) control problem with the aim of tracking a desired output trajectory. The objective (or cost) function is assumed to be quadratic. Although the optimal control of the input-output model in section 4.3 is formulated in a deterministic environment, the generalization to stochastic optimal control with future uncertainties and measurement errors is proposed in the subsequent sections.

Optimal consumption policies for the dynamic Leontief model using Pontryagin's maximum principle is the subject of section 4.4. Using the Hamiltonian function, we have obtained the matrix Riccati equation and the tracking equation when the objective function is quadratic. It is shown in this section that by solving simultaneously the matrix Riccati and tracking equations together with the boundary conditions one can derive, using an iterative technique, the optimal consumption policies and the resulting optimal output trajectory for the entire planning period.

Bellman's dynamic programming has been used in section 4.5 to obtain the optimal consumption policies for the same model introduced is section 4.4. However, for mathematical convenience, we have first reduced the optimal control of Leontief models in tracking format to a regulator problem. This will markedly facilitate the applications of Bellman's principle of optimality to obtain the functional recurrence equation for Leontief models. By using a forward and backward solution, we have obtained optimal consumption policies and the ensuing optimal output path.

Sections 4.6, 4.7 and 4.8 extend deterministic optimal control of the dynamic Leontief model to the case where there exists *i*) a deterministic (and stochastic) substitution system, *ii*) uncertainties about the future state of the Leontief model resulting from the current decisions on substitute activities, *iii*) measurement errors in the substitution system and *iv*) a stochastic substitution system together with the stochastic Leontief system dynamics.

It is well-known that the optimal solution to an input-output model is not unique since the set of different policies on alternative substitute activities will result in different output trajectories. This implies that the values of alternative activities should be chosen in such a way as to minimise a cost function. Arrow (1951), Samuelson (1951) and Morishima (1965) have solved, for the first time, the Leontief substitution model with a linear cost function using the method of linear programming. In this chapter, I have solved the extended version of this problem where the objective function is non-linear and the system dynamics is stochastic. More importantly, I have solved the stochastic dynamic Leontief model when there exists a stochastic substitution system.

The generalized stochastic dynamic programming functional recurrence equation, developed by Jacobs (1977), has been used to obtain the optimal substitute activities. This optimal solution is then regarded as the input to the greater problem of stochastic control of a linear Leontief dynamic system which itself is assumed to be perturbed by a disturbance term affecting its future state. A quadratic cost function is used and the dynamic Leontief model with substitution is then solved as a tracking problem in stochastic control theory. Further generalization are made by using *i*) a non-linear (non-quadratic) cost function, *ii*) non-linear system dynamics and *iii*) measurement errors in substitute activities.

Optimal control formulation of substitute activities is presented in section 4.6. We have assumed in this section that a policy taken in a particular period can be used as an argument in the price behaviour next period since such policies influence the demand pattern for different commodities in that period and the supply pattern of such goods. Our policy on substitute activities will not, therefore, be independent of the current policies. It follows that an optimising solution is necessary over the entire period of planning substitute activities. We have further assumed the presence of uncertainties in future values of the price vector when a policy is chosen in substitute activities. The existence of measurement errors is assumed in the substitution system. For the class of stochastic substitution systems with uncertainties about its future state and with measurement errors a general non-linear cost function is assumed.

The dynamic programming solution of the Leontief system of substitute activities is presented in section 4.7. We have assumed in this section that, due to measurement errors, the current price vector in a Leontief substitution system is described by a probability density function. This function will be conditional upon the past measurements of price vector as well as on the past policies chosen on substitute activities. This conditional probability density function is assumed to be updated at each stage in the optimization process by Bayes' rule. Thus we are faced with two separate functions for updating conditional probability for the price vector together with generating optimal policies for the Leontief substitute system.

By using Bellman's principle of optimality, we have obtained the dynamic programming recurrence equation which determines the way in which optimal policies for the Leontief substitution system can be produced. In doing so, we have used Åstrom's fundamental lemma of stochastic control theory as generalized by Jacobs (1977) for non-linear systems. The functional recurrence equation formulates the optimal substitute activities at any time t, in order to achieve the minimum value of the total cost of production over the entire planning horizon.

Section 4.8 deals with optimal stochastic control of a dynamic Leontief model with future uncertainties and stochastic substitution. In this section, a quadratic objective function is used in which any deviations of the total output vector from its desired trajectory are to be minimised. A stochastic substitution system with non-linear dynamics and uncertainties about both the present and future states is incorporated in the stochastic dynamic Leontief model. The optimal sequence of policies for the substitution system obtained in section 4.7 is then regarded as an input to the controller of the stochastic dynamic input-output model which updates the Leontief matrix at each planning period. Again, by using Åstrom's fundamental lemma of stochastic control theory, we have obtained the matrix Riccati equation for stochastic dynamic Leontief model which together with boundary conditions determine the optimal feedback control law for the final consumption path.

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